Hypersequent calculi for non classical logics

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Non classical logics

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A central task of logic in computer science is to provide *calculi* for large families of *non-classical logics*. 
Uniform and Analytic Calculi

Uniform calculi facilitate the switch from one logic to another, deepening the understanding of the relations between them. Analytic calculi are a basic prerequisite for developing automated reasoning methods. Moreover they may help to resolve such meta-logical issues as decidability, complexity, interpolation and admissibility of rules.
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Sequent Calculi

G. Gentzen “Untersuchungen über das logische Schliessen I, II”. Mathematische Zeitschrift 1934
Sequent Calculi

Sequents

\[ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \]
Sequent Calculi

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Intuitively a sequent is understood as “the conjunction of \( A_1, \ldots, A_n \) implies the disjunction of \( B_1, \ldots, B_m \)”
Sequent Calculi

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Axioms

E.g., \( A \vdash A \)

Rules

- Logical
- Structural

\[ \Gamma, A \vdash \Delta \quad \Gamma' \vdash A, \Delta' \]

\[ \Gamma, \Gamma' \vdash \Delta, \Delta' \quad \text{(cut)} \]
Ex: LK calculus for CL

(sequents = multisets of formulas)

\[ A \vdash A \]

\[
\text{(cut)} \quad \frac{\Gamma_1 \vdash A, \Delta \quad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B, \Delta}
\]

\[ (w, l) \quad \frac{\Gamma \vdash C, \Delta}{\Gamma, A \vdash C, \Delta} \]

\[ (w, r) \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash C, \Delta} \]

\[ (c, l) \quad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \]

\[ (c, r) \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \]

\[ (\rightarrow, r) \quad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \]

\[ (\rightarrow, l) \quad \frac{\Gamma \vdash A, \Delta \quad B, \Gamma \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \]
Summary
Summary

1. Introduction to Hypersequents

2. Adding quantifiers

3. Advanced Topics
Summary

1. Introduction to Hypersequents
   - Examples of HC: Intuitionistic and Classical Logic, LQ, Logics of bounded cardinality Kripke Models
2. Adding quantifiers

3. Advanced Topics
Summary

1. Introduction to Hypersequents
   **Examples** of HC: Intuitionistic and Classical Logic, LQ, Logics of bounded cardinality Kripke Models
2. Adding quantifiers
   **Example:** Gödel logics (propositional, first order and propositionally quantified)
3. Advanced Topics
Summary

1. Introduction to Hypersequents
   Examples of HC: Intuitionistic and Classical Logic, LQ, Logics of bounded cardinality Kripke Models

2. Adding quantifiers
   Example: Gödel logics (propositional, first order and propositionally quantified)

3. Advanced Topics
   - cut-elimination
   - automated generation of hypersequent calculi (Examples: basic fuzzy logics and GIL)
   - variants of the hypersequent framework (Examples: Łukasiewicz logic and Logics of bounded depth Kripke Models)
Hypersequent Calculi

1

1

1

j

j

n

where, for all

i = 1;::: n;

i

is an ordinary

sequent.

i

is called a

component

of the hypersequent.

The symbol "j" is intended to denote disjunction at

the meta-level.

The above hypersequent is interpreted as

G

_ ::::

_ G

n

where

G

i

is the interpretation of the sequent

i

.
Hypersequent Calculi

(see also G. Pottinger “Uniform, cut-free formulation of T,S_4 and S_5, (abstract)”. J. of Symbolic Logic. 1983.)
Hypersequent Calculi

are a simple and natural generalization of Sequent Calculi. A hypersequent is

\[ \Gamma_1 \vdash \Pi_1 | \ldots | \Gamma_n \vdash \Pi_n \]

where, for all \( i = 1, \ldots, n \), \( \Gamma_i \vdash \Pi_i \) is an ordinary sequent.
Hypersequent Calculi

\[ \Gamma_1 \vdash \Pi_1 \mid \ldots \mid \Gamma_n \vdash \Pi_n \]

where, for all \( i = 1, \ldots, n \), \( \Gamma_i \vdash \Pi_i \) is an ordinary sequent. 
\( \Gamma_i \vdash \Pi_i \) is called a component of the hypersequent.
Hypersequent Calculi

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\( \Gamma_i \vdash \Pi_i \) is called a component of the hypersequent. The symbol “\( \mid \)” is intended to denote disjunction at the meta-level.
Hypersequent Calculi

\[ \Gamma_1 \vdash \Pi_1 \mid \ldots \mid \Gamma_n \vdash \Pi_n \]

where, for all \( i = 1, \ldots, n \), \( \Gamma_i \vdash \Pi_i \) is an ordinary sequent.
\( \Gamma_i \vdash \Pi_i \) is called a component of the hypersequent.
The symbol “\( \mid \)” is intended to denote disjunction at the meta-level.
The above hypersequent is interpreted as

\[ G_1 \lor \ldots \lor G_n \]

where \( G_i \) is the interpretation of the sequent \( \Gamma_i \vdash \Pi_i \)
Hypersequent Calculi

- general framework
Hypersequent Calculi

- general framework
- independent of any particular semantics
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- the rules for connectives are *standard*: the difference between the hypersequent calculi for various logics is in the set of their structural rules
Hypersequent Calculi

- general framework
- independent of any particular semantics
- the rules for connectives are *standard*: the difference between the hypersequent calculi for various logics is in the set of their structural rules (Facilitate a better understanding and construction of the logics constructed within the framework.)
Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules
Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules

- Logical
- Structural
- Cut
Hypersequent Calculi

**Axioms**  E.g., \( A \vdash A \)

**Rules**  Logical & Cut
Hypersequent Calculi

**Axioms** E.g., \( A \vdash A \)

**Rules**

**Logical & Cut**

E.g,

\[
\frac{G \mid \Gamma \vdash A, \Delta \quad G' \mid B, \Gamma \vdash \Delta}{G \mid G' \mid \Gamma, A \rightarrow B \vdash \Delta} \quad (\rightarrow, l)
\]
Hypersequent Calculi

Axioms  E.g.,  \( A \vdash A \)

Rules

Logical & Cut

Structural

- Internal
- External

\[
\frac{G}{G \mid \Gamma \vdash \Delta} \quad (EW) \\
\frac{G \mid \Gamma' \vdash \Delta' \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta \mid \Gamma' \vdash \Delta'} \quad (EE) \\
\frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta} \quad (EC')
\]
(Hyper)sequent Calculus for IL

\[ A \vdash A \]

\[(\text{cut}) \quad \frac{\Gamma_1 \vdash A \quad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B} \]

\[(w) \quad \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \]

\[(c) \quad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \]

\[(\rightarrow, r) \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \]

\[(\rightarrow, l) \quad \frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{\Gamma, A \rightarrow B \vdash C} \]
(Hyper)sequent Calculus for IL

\[ A \vdash A \]

(cut) \[
\begin{align*}
G | \Gamma_1 \vdash A & \quad G' | A, \Gamma_2 \vdash B \\
G | G' | \Gamma_1, \Gamma_2 \vdash B
\end{align*}
\]

\( (w) \) \[
\frac{G | \Gamma \vdash C}{G | \Gamma, A \vdash C}
\]

\( (c) \) \[
\frac{G | \Gamma, A, A \vdash C}{G | \Gamma, A \vdash C}
\]

\( (EE), (EW) \)

\( (EC') \)

\( (\to, r) \) \[
\frac{G | \Gamma, A \vdash B}{G | \Gamma \vdash A \to B}
\]

\( (\to, l) \) \[
\frac{G | \Gamma \vdash A \quad G' | B, \Gamma \vdash C}{G | G' | \Gamma, A \to B \vdash C}
\]
A hypersequent

\[ \Gamma_1 \vdash A_1 \mid \ldots \mid \Gamma_n \vdash A_n \]

is provable in the hypersequent calculus for IL if and only if there exists \( i \in \{1, \ldots, n\} \) such that LJ proves

\[ \Gamma_i \vdash A_i \]
Some Structural Rules

\[
\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash \mid \Gamma' \vdash A} \quad (cl)
\]

\[
\frac{G \mid \Gamma, \Gamma' \vdash}{G|\Gamma \vdash \mid \Gamma' \vdash} \quad (lq)
\]

\[
\frac{G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma'_1 \vdash A'}{G|G'|\Gamma, \Gamma'_1 \vdash A|\Gamma', \Gamma_1 \vdash A'} \quad (com)
\]
Avron suggested that a hypersequent can be thought of as a multiprocess. (A. Avron “Hypersequents, Logical Consequence and Intermediate Logics for Concurrency”. Annals of Mathematics and Artificial Intelligence. 1991) 

Fermüller characterized hypersequent calculi by (suitable) games of communicating parallel dialogues (C.G. Fermüller “Parallel Dialogue Games and Hypersequents for Intermediate Logics”. Tableaux 2003)
Hypersequents and Parallelism

- Avron suggested that a hypersequent can be thought of as a multiprocess. 
Hypersequents and Parallelism

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\[
G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma'_1 \vdash A' \\
\frac{G|G'|\Gamma, \Gamma'_1 \vdash A|\Gamma', \Gamma_1 \vdash A'}{G'} (com)
\]

Fermüller characterized hypersequent calculi by (suitable) games of communicating parallel dialogues
Hypersequents and Parallelism

- Avron suggested that a hypersequent can be thought of as a multiprocess. (A. Avron “Hypersequents, Logical Consequence and Intermediate Logics for Concurrency”. Annals of Mathematics and Artificial Intelligence. 1991)

- Fermüller characterized hypersequent calculi by (suitable) games of communicating parallel dialogues (C.G. Fermüller “Parallel Dialogue Games and Hypersequents for Intermediate Logics”. Tableaux 2003)
Example: CL
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\[ A \vdash A \]

\[ (w, l) \quad \frac{G|\Gamma \vdash C, \Delta}{G|\Gamma, A \vdash C, \Delta} \]

\[ (c, l) \quad \frac{G|\Gamma, A, A \vdash C}{G|\Gamma, A \vdash C} \]

\[ (EE), (EW) \quad \frac{G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \to B, \Delta} \]

\[ (c, r) \quad \frac{G|\Gamma \vdash A, A, \Delta}{G|\Gamma \vdash A, \Delta} \]

\[ (\to, r) \quad \frac{G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \to B, \Delta} \]

\[ (w, r) \quad \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash C, \Delta} \]

\[ (\to, l) \quad \frac{G|\Gamma \vdash A, \Delta}{G|G'|\Gamma, A \to B \vdash \Delta} \]

\[ (EE') \quad \frac{G|\Gamma \vdash A, \Delta}{G|\Gamma \vdash A, \Delta} \]

\[ (EW') \quad \frac{G'|B, \Gamma \vdash \Delta}{G|G'|\Gamma, A \to B \vdash \Delta} \]
Example: CL

Hypersequent calculus for IL +

\[
\frac{G | \Gamma, \Gamma' \vdash A}{G | \Gamma \vdash |\Gamma' \vdash A} \quad (cl)
\]

= (cut-free) Hypersequent calculus for Classical Logic

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\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash |\Gamma' \vdash A} \quad (cl)
\]

= (cut-free) Hypersequent calculus for Classical Logic

Idea behind the rule: let \( \Gamma' = \emptyset \) and \( \Gamma = A \).

\[
\frac{G|A \vdash A}{G|A \vdash | \vdash A} \quad (cl)
\]

The \((cl)\) rule performs a case-analysis, namely it says that for every formula \( A \) and evaluation \( \nu \), \( \nu(A) = 0 \) or \( \nu(A) = 1 \).
Example: CL

Hypersequent calculus for IL +

\[
\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash |\Gamma' \vdash A}\quad (cl)
\]

= (cut-free) Hypersequent calculus for Classical Logic

Soundness & Completeness

Hilbert system: set of axiom schemes together with one rule of inference: modus ponens

\[
\frac{A}{A \rightarrow B} \quad A \rightarrow B
\]
Example: CL

Hypersequent calculus for IL +

\[
\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash \Gamma' \vdash A} (cl)
\]

= (cut-free) Hypersequent calculus for Classical Logic

Soundness

We prove that the interpretation of axioms (i.e. \( A \rightarrow A \)) is provable in the Hilbert system for CL and that for each rule that whenever the Hilbert system for CL derives the interpretations of its premises, it derives the interpretation of its conclusion too. E.g., For \((cl)\): if \( G \lor ((\land \Gamma \land \land \Gamma') \rightarrow A) \) is provable in the H. system for CL, so is \( G \lor \neg \land \Gamma \lor (\land \Gamma') \rightarrow A. \)
Example: CL

Hypersequent calculus for IL +

\[
\frac{G \mid \Gamma, \Gamma' \vdash A}{G \mid \Gamma \vdash \Gamma' \vdash A} \quad (cl)
\]

= (cut-free) Hypersequent calculus for Classical Logic

Completeness

Modus Ponens corresponds to the derivability of \( A, A \rightarrow B \vdash B \) and the cut rule. It thus suffices to show that all the axioms of the Hilbert system for CL are derivable in the calculus.
Example: CL

Hypersequent calculus for IL +

\[ \frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash |\Gamma' \vdash A} \]  \hspace{1cm} (cl)

= (cut-free) Hypersequent calculus for Classical Logic

\[ \frac{A \vdash A}{A \vdash | \vdash A} \]  \hspace{1cm} (cl)

\[ \frac{\vdash \neg A | \vdash A}{\neg \neg A \vdash | \vdash A} \]  \hspace{1cm} (\neg, l)

\[ \frac{\neg \neg A \vdash A | \neg \neg A \vdash A}{\neg \neg A \vdash A} \]  \hspace{1cm} 2x(w)

\[ \frac{\neg \neg A \vdash A | \neg \neg A \vdash A}{\neg \neg A \vdash A} \]  \hspace{1cm} (EC)

\[ \frac{\vdash \neg \neg A \rightarrow A}{\vdash \neg \neg A \rightarrow A} \]  \hspace{1cm} (\rightarrow, r)
Example: LQ

LQ is the intermediate logic semantically characterized by the class of all finite and rooted posets with a single final element.
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Example: LQ

LQ is the intermediate logic semantically characterized by the class of all finite and rooted posets with a single final element. A Hilbert-style axiomatization for LQ is obtained by adding to that of IL, i.e., e.g.

\[ \top \rightarrow A \quad A \rightarrow (B \rightarrow A) \]
\[ (A \land B) \rightarrow A \quad (A \land B) \rightarrow B \]
\[ A \rightarrow (A \lor B) \quad B \rightarrow (A \lor B) \]
\[ (A \rightarrow (B \rightarrow C')) \rightarrow (B \rightarrow (A \rightarrow C')) \]
\[ (A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B)) \]
\[ A \rightarrow (B \rightarrow (A \land B)) \]
\[ (A \rightarrow B) \rightarrow ((C \rightarrow B) \rightarrow ((A \lor C') \rightarrow B)) \]
\[ (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B) \]

V. A. Jankov "The calculus of the weak "law of excluded middle". Mathematics of the USSR. 1968"
Example: LQ

LQ is the *intermediate logic* semantically characterized by the class of all finite and rooted posets with a single final element. A Hilbert-style axiomatization for LQ is obtained by adding to that of IL axiom

\[ \neg A \lor \neg \neg A \]

(V.A. Jankov "The calculus of the weak “law of excluded middle”". Mathematics of the USSR. 1968)
A Hypersequent calculus for LQ

Sequent Formulation:
A Hypersequent calculus for LQ

Hypersequent calculus for IL +

\[
\frac{G | \Gamma, \Gamma' \vdash}{G | \Gamma \vdash \Gamma' \vdash} (lq)
\]

= (cut-free) Hypersequent calculus for LQ

A Hypersequent calculus for LQ

Hypersequent calculus for IL +

\[
\frac{G \mid \Gamma, \Gamma' \vdash}{G \mid \Gamma \vdash \mid \Gamma' \vdash} \quad (lq)
\]

= (cut-free) Hypersequent calculus for LQ

\[
\frac{A \vdash A}{A, \neg A \vdash} \quad (\neg, l)
\]

\[
\frac{A \vdash \neg A \vdash}{A \vdash \vdash \neg A} \quad (lq)
\]

\[
\frac{A \vdash \vdash \neg A}{A \vdash \vdash \neg \neg A} \quad (\neg, r)
\]

\[
\frac{A \vdash \neg \neg A}{\neg \neg A \vdash \neg A \vdash} \quad (\neg, r)
\]

\[
\frac{\neg \neg A \vdash \neg A \vdash \neg \neg A}{\neg \neg A \vdash \neg A \lor \neg \neg A} \quad (\lor, r)
\]

\[
\frac{\neg A \lor \neg \neg A \vdash \neg A \lor \neg \neg A}{\neg \neg A \vdash \neg A \lor \neg \neg A} \quad (\lor, r)
\]

\[
\frac{\neg A \lor \neg \neg A \lor \neg A \lor \neg \neg A}{\neg A \lor \neg \neg A \lor \neg A \lor \neg \neg A} \quad (EC)
\]
Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most $k$ nodes.
Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most $k$ nodes. A Hilbert style axiomatization is given by

$$IL + \{ p_0 \lor (p_0 \rightarrow p_1) \lor \ldots \lor (p_0 \land \ldots \land p_{k-1} \rightarrow p_k) \}$$
Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most $k$ nodes. Particular cases: $k = 1$ Classical Logic, $k = 2$ SM logic.
Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most $k$ nodes.

Hypersequent Calculus
(for $k \geq 1$) Hypersequent calculus for IL +

\[
\begin{align*}
&\ldots \quad G_{i,j} \mid \Gamma_i, \Gamma_j \vdash A_i \quad \ldots \\
G_{0,1} \mid \ldots \mid G_{k-1,k} \mid \Gamma_0 \vdash A_0 \mid \ldots \mid \Gamma_{k-1} \vdash A_{k-1} \mid \Gamma_k \vdash \end{align*}
\]

for every $i, j$ such that $0 \leq i \leq k - 1$ and $i + 1 \leq j \leq k$.

(*, M. Ferrari. "Hypersequent calculi for some intermediate logics with bounded Kripke models". J. of Logic and Computation. 2001)
Summary

1. Introduction to Hypersequents

2. Adding quantifiers
   
   Example: Gödel logics (propositional, first order and propositionally quantified)

3. Advanced Topics
Adding Quantifiers

Two different forms of quantification:
- First-order quantifiers (universal and existential quantification over object variables)
- Propositional quantifiers (universal and existential quantification over propositions)

Adding Quantifiers

Two different forms of quantification:

• first-order quantifiers

• propositional quantifiers
Adding Quantifiers

Two different forms of quantification:

- first-order quantifiers (universal and existential quantification over object variables)
- propositional quantifiers

Adding Quantifiers

Two different forms of quantification:

- first-order quantifiers
  (universal and existential quantification over object variables)

- propositional quantifiers
  (universal and existential quantification over propositions)
Adding Quantifiers

Two different forms of quantification:

- first-order quantifiers
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  (universal and existential quantification over propositions)

Gödel logic(s)

Gödel '33

related to relevance logics (Dunn and Meyer '71)

employed to investigate the provability logic of Heyting arithmetic (Visser '82)

used to model strong equivalence between logic programs (Lifschitz et al. 2002)

one of the main formalizations of fuzzy logic (Hajek '98)
Gödel logic(s)

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Gödel ’33
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Gödel logic(s)

- related to relevance logics (Dunn and Meyer ’71)
- employed to investigate the provability logic of Heyting arithmetic (Visser ’82)
- used to model strong equivalence between logic programs (Lifschitz et al.2002)

\[ sm(P_1 \bigcup P) = sm(P_2 \bigcup P) \quad \forall P \]
Gödel logic(s)

- related to relevance logics (Dunn and Meyer ’71)
- employed to investigate the provability logic of Heyting arithmetic (Visser ’82)
- used to model strong equivalence between logic programs (Lifschitz et al.2002)
- one of the main formalizations of fuzzy logic (Hajek’98)
Gödel logic $G_\infty$

$G_\infty$ is characterized by the class of all rooted linearly ordered Kripke models.
Gödel logic $G_{\infty}$

$G_{\infty}$ is characterized by the class of all rooted linearly ordered Kripke models.

Axiomatization

$$G_{\infty} = IL + (A \rightarrow B) \lor (B \rightarrow A)$$

(M. Dummett ”A Propositional Logic with Denumerable Matrix” J. of Symbolic Logic. 1959)
Gödel logic $G_\infty$

$G_\infty$ is characterized by the class of all rooted linearly ordered Kripke models.

**Axiomatization**

$$G_\infty = IL + (A \rightarrow B) \lor (B \rightarrow A)$$

**Many-valued Semantics**

$v : \text{Propositions} \rightarrow [0, 1]$

$v(A \land B) = \min\{v(A), v(B)\}$

$v(A \lor B) = \max\{v(A), v(B)\}$

$v(A \rightarrow B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases}$

$v(\bot) = 0$
Hypersequent Calculus for $G_\infty$
Hypersequent Calculus for $G_{\infty}$

Cut free Sequent calculus
O. Sonobo. ”A Gentzen-type formulation for some intermediate propositional logics”. J. of Tsuda College. 1975
Hypersequent Calculus for $G_{\infty}$

Hypersequent Calculus for Intuitionistic Logic +

\[
\frac{G \mid \Gamma, \Gamma' \vdash A \quad G' \mid \Gamma_1, \Gamma'_1 \vdash A'}{G \mid G' \mid \Gamma, \Gamma_1 \vdash A \mid \Gamma', \Gamma'_1 \vdash A'} \quad (com)
\]

Hypersequent Calculus for $G_\infty$

Hypersequent Calculus for Intuitionistic Logic +

$$G \mid \Gamma, \Gamma' \vdash A \quad G' \mid \Gamma_1, \Gamma'_1 \vdash A'$$

$$G \mid G' \mid \Gamma, \Gamma_1 \vdash A \mid \Gamma', \Gamma'_1 \vdash A' \quad (com)$$

Example (completeness)

$$B \vdash B \quad A \vdash A$$

$$A \vdash B \mid B \vdash A \quad (com)$$

$$A \vdash B \mid \vdash B \rightarrow A \quad (\rightarrow, r)$$

$$\vdash A \rightarrow B \mid \vdash B \rightarrow A \quad (\rightarrow, r)$$

$$\vdash A \rightarrow B \mid \vdash (A \rightarrow B) \lor (B \rightarrow A) \quad (\lor_i, r)$$

$$\vdash (A \rightarrow B) \lor (B \rightarrow A) \mid \vdash (A \rightarrow B) \lor (B \rightarrow A) \quad (\lor_i, r)$$

$$\vdash (A \rightarrow B) \lor (B \rightarrow A) \quad (BC)$$
**Gödel logic** $G_{k+1}$

Infinite valued Gödel logic is characterized by the class of all rooted linearly ordered Kripke models

**Axiomatization**

$$G_\infty = IL + (A \rightarrow B) \lor (B \rightarrow A)$$

**Many-valued Semantics**

$v : \text{Propositions} \rightarrow [0, 1]$

\[
v(A \land B) = \min\{v(A), v(B)\}
v(A \lor B) = \max\{v(A), v(B)\}
v(A \rightarrow B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases} \quad v(\bot) = 0
\]
Gödel logic $G_{k+1}$

$G_{k+1}$ is characterized by the class of all rooted linearly ordered Kripke models with at most $k$ worlds

**Axiomatization**

$$G_{\infty} = IL + (A \rightarrow B) \lor (B \rightarrow A)$$

$$+ A_1 \lor (A_1 \rightarrow A_2) \lor \ldots \lor (A_1 \land \ldots \land A_k \rightarrow A_{k+1})$$

**Many-valued Semantics**

$v : \text{Propositions} \rightarrow [0, 1] \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$

$v(A \land B) = \min\{v(A), v(B)\}$

$v(A \lor B) = \max\{v(A), v(B)\}$

$v(A \rightarrow B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases}$

$v(\bot) = 0$
Hypersequent Calculi for $G_{k+1}$

(*, M. Ferrari Hypersequent calculi for some intermediate logics with bounded Kripke models". J. of Logic and Computation. 2001)

Hypersequent Calculus for Intuitionistic Logic +
Hypersequent Calculi for $G_{k+1}$

Hypersequent Calculus for Intuitionistic Logic +

\[
\frac{G | \Gamma, \Gamma' \vdash A \quad G' | \Gamma_1, \Gamma'_1 \vdash A'}{G | G' | \Gamma, \Gamma_1 \vdash A | \Gamma', \Gamma'_1 \vdash A'} \quad (\text{com})
\]

\[+\]

\[
\vdots \quad G_{i,j} | \Gamma_i, \Gamma_j \vdash A_i \quad \ldots
\]

\[
G_{0,1} | \ldots | G_{k-1,k} | \Gamma_0 \vdash A_0 | \ldots | \Gamma_{k-1} \vdash A_{k-1} | \Gamma_k \vdash \]

for every $i, j$ such that $0 \leq i \leq k - 1$ and $i + 1 \leq j \leq k$. 
Hypersequent Calculi for $G_{k+1}$

Hypersequent Calculus for Intuitionistic Logic +

$$
\begin{array}{c}
G_1 | \Gamma_1, \Gamma_2 \vdash A_1 \quad \ldots \quad G_k | \Gamma_k, \Gamma_{k+1} \vdash A_k \\
\hline
G_1 | \ldots | G_k | \Gamma_1 \vdash A_1 | \ldots | \Gamma_k \vdash A_k | \Gamma_{k+1} \vdash
\end{array}
$$
Example: HC for $G_3$

Hypersequent Calculus for Intuitionistic Logic +

\[
\frac{G_1 \mid \Gamma_1, \Gamma_2 \vdash A_1 \quad G_2 \mid \Gamma_2, \Gamma_3 \vdash A_2}{G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid \Gamma_3 \vdash} (G_3)
\]
Example: HC for $G_3$

Hypersequent Calculus for Intuitionistic Logic +

\[
G_1 \mid \Gamma_1, \Gamma_2 \vdash A_1 \quad G_2 \mid \Gamma_2, \Gamma_3 \vdash A_2 \\
G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid \Gamma_3 \vdash \\ (G_3)
\]

Ex: \((P := A_1 \lor (A_1 \rightarrow A_2) \lor (A_1 \land A_2 \rightarrow A_3))\)

\[
\frac{A_2 \vdash A_2}{A_1 \land A_2 \vdash A_2} \quad (\land,1)
\]

\[
\frac{A_1 \vdash A_1 \quad A_1, A_1 \land A_2 \vdash A_2}{A_1 \land A_2 \vdash A_2} \quad (w)
\]

\[
\frac{A_1 \vdash A_1 \quad \vdash A_1 \mid \vdash A_2 \mid A_1 \land A_2 \vdash}{\vdash A_1 \mid A_1 \vdash A_2 \mid A_1 \land A_2 \vdash} \quad (G_3)
\]

\[
\frac{\vdash A_1 \mid \vdash A_1 \vdash A_2 \mid A_1 \land A_2 \vdash}{\vdash A_1 \mid \vdash A_1 \rightarrow A_2 \mid \vdash A_1 \land A_2 \rightarrow A_3} \quad \quad 2x(\rightarrow,r)
\]

\[
\frac{\vdash P \mid \vdash P \mid \vdash P}{\vdash P} \quad 3x(\lor,r)
\] 

\[
2x(EC)
\]

Tableaux’03 – p.25/63
First Order Gödel logic

Intuitionistic Fuzzy Logic (Takeuti and Titani. JSL. 1984)
First Order Gödel logic

is characterized by the class of all rooted linearly ordered Kripke models with *constant domains*.
First Order Gödel logic

is characterized by the class of all rooted linearly ordered Kripke models with \textit{constant domains}.

Many valued semantics
An interpretation $I = (\text{domain } D, \text{ valuation function } v_I) (P^n \text{ mapped into } D^n \rightarrow [0, 1])$.

\[
v_I((\forall x)A(x)) = \inf \{v'_I(A(x))\}
\]
\[
v_I((\exists x)A(x)) = \sup \{v'_I A(x)\}
\]

(where $v'_I$ is exactly as $v_I$ with the possible exception of the domain element assigned to $x$).
First Order Gödel logic

is characterized by the class of all rooted linearly ordered Kripke models with constant domains.

Many valued semantics

An interpretation \( I = (\text{domain } D, \text{ valuation function } \nu_I) \) (\( P^n \) mapped into \( D^n \to [0, 1] \)).

\[
\nu_I((\forall x)A(x)) = \inf \{ \nu_{I'}(A(x)) \}
\]

\[
\nu_I((\exists x)A(x)) = \sup \{ \nu_{I'}(A(x)) \}
\]

(where \( \nu_{I'} \) is exactly as \( \nu_I \) with the possible exception of the domain element assigned to \( x \)).

\[
G_\infty = IL + (A \to B) \lor (B \to A)
\]

+ Ax for quant. + \( \forall x (A(x) \lor B) \to (\forall x A(x)) \lor B \)
First Order Gödel logic

is characterized by the class of all rooted linearly ordered Kripke models with constant domains.

\[
G_\infty = IL + (A \rightarrow B) \lor (B \rightarrow A)
\]

\[+ \text{Ax for quant. } + \forall x (A(x) \lor B) \rightarrow (\forall x A(x)) \lor B\]

\[? + ? \text{ (Takeuti and Titani)}\]

\[
\frac{\Gamma \vdash C \lor (A \rightarrow p) \lor (p \rightarrow B)}{\Gamma \vdash C \lor (A \rightarrow B)}
\]

where \(p\) is does not occur in the conclusion.
Hypersequent Calculus for FO Gödel Logic

M. Baaz and R. Zach "Hypersequents and the proof theory of intuitionistic fuzzy logic". Proc. of CSL’2000

Hypersequent Calculus for propositional Gödel Logic +
Hypersequent Calculus for FO Gödel Logic

Hypersequent Calculus for propositional Gödel Logic

\[ \frac{G \mid A(t), \Gamma \vdash B}{G \mid (\forall x) A(x), \Gamma \vdash B} \quad (\forall, l) \]

\[ \frac{G \mid A(a), \Gamma \vdash B}{G \mid (\exists x) A(x), \Gamma \vdash B} \quad (\exists, l) \]

\[ \frac{G \mid \Gamma \vdash A(a)}{G \mid \Gamma \vdash (\forall x) A(x)} \quad (\forall, r) \]

\[ \frac{G \mid \Gamma \vdash A(t)}{G \mid \Gamma \vdash (\exists x) A(x)} \quad (\exists, r) \]
Hypersequent Calculus for propositional Gödel Logic

\[ \begin{align*}
G \mid A(t), \Gamma \vdash B & \rightarrow G \mid (\forall x)A(x), \Gamma \vdash B \quad (\forall, l) \\
G \mid \Gamma \vdash A(a) & \rightarrow G \mid \Gamma \vdash (\forall x)A(x) \quad (\forall, r) \\
G \mid A(a), \Gamma \vdash B & \rightarrow G \mid (\exists x)A(x), \Gamma \vdash B \quad (\exists, l) \\
G \mid \Gamma \vdash A(t) & \rightarrow G \mid \Gamma \vdash (\exists x)A(x) \quad (\exists, r)
\end{align*} \]

where in \((\forall, r)\) and \((\exists, l)\) the free variable \(a\) must not occur in the lower hypersequent.
HC for FO Gödel Logic

Ex.

\[
\begin{align*}
A(a) \vdash A(a) & \quad B \vdash B \\
A(a) \vdash A(a) & \quad B \vdash A(a) \quad | \quad A(a) \vdash B & \quad (\text{com}) & \quad B \vdash B \\
A(a) \lor B \vdash A(a) & \quad | \quad A(a) \lor B \vdash B \\
(\forall x)(A(x) \lor B) \vdash A(a) & \quad | \quad (\forall x)(A(x) \lor B) \vdash B & \quad (\forall,x)l \\
(\forall x)(A(x) \lor B) \vdash (\forall x)A(x) & \quad | \quad (\forall x)(A(x) \lor B) \vdash (\forall x)A(x) & \quad (\forall,x)r \\
(\forall x)(A(x) \lor B) \vdash (\forall x)A(x) \lor B & \quad | \quad (\forall x)(A(x) \lor B) \vdash (\forall x)A(x) \\
(\forall x)(A(x) \lor B) \vdash (\forall x)A(x) \lor B & \quad (\rightarrow,r) \\
\vdash (\forall x)(A(x) \lor B) \rightarrow ((\forall x)A(x) \lor B) &
\end{align*}
\]
Midsequent Theorem

In Gentzen’s **LK** — as a consequence of cut-elimination — a separation between propositional and quantificational inferences can be achieved in deriving a prenex sequent (**midsequent theorem**). See, e.g. G. Takeuti. Proof Theory. 1987
Midsequent Theorem

In Gentzen’s LK — as a consequence of cut-elimination — a separation between propositional and quantificational inferences can be achieved in deriving a prenex sequent (midsequent theorem).

(Sketch of Proof)
Proceeds by induction on the order of a derivation. From

quantifier rule

____________

logical rule

To

logical rule

____________

quantifier rule

Example:

\[ \forall x (A(x) \land \exists y (B(y) \land C(x,y))) \]

\[ \exists y (B(y) \land C(x,y)) \]
Midsequent Theorem

In Gentzen’s **LK** — as a consequence of cut-elimination — a separation between propositional and quantificational inferences can be achieved in deriving a prenex sequent (midsequent theorem). This result does not hold for **LJ**.

E.g.

\[
\begin{align*}
\Gamma, Y &\vdash A(a) \\
\Gamma, Y &\vdash (\forall x)A(x) \\
\Gamma &\vdash (\forall x)A(x) \\
\Gamma, X &\vdash A(b) \\
\Gamma, X &\vdash (\forall x)A(x) \\
\Gamma &\vdash (\forall x)A(x) \\
\Gamma, X \lor Y &\vdash (\forall x)A(x)
\end{align*}
\]
Any derivation in the HC for FO Gödel Logic of a prenex hypersequent can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.

(Sketch of Proof)

Proceeds by induction on the order of a derivation.

Note that the following rule is derivable in the HC for FO Gödel Logic.

\[
G_j \vdash A; \vdash C_1 \land G_j B; \vdash C_2 \land (\_0; l)
\]
Any derivation in the HC for FO Gödel Logic of a prenex hypersequent can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.
Any derivation in the HC for FO Gödel Logic of a prenex hypersequent can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.

*(Sketch of Proof)*

Proceeds by induction on the *order* of a derivation. Note that the following rule is derivable in the HC for FO Gödel Logic.

\[
\begin{array}{c}
G | A, \Gamma \vdash C_1 \\
G | B, \Gamma \vdash C_2
\end{array}
\rightarrow
G | A \lor B, \Gamma \vdash C_1 | A \lor B, \Gamma \vdash C_2
\]

\((\lor', l)\)
Mid(hyper)sequent Theorem — Sketch of Proof

We replace all the applications of \((\lor, l)\) by applications of \((\lor', l)\).
Mid(hyper)sequent Theorem – Sketch of Proof

We replace all the applications of \((\forall, l)\) by applications of \((\forall', l)\). Therefore

\[
\Gamma, Y \vdash A(a) \quad \Gamma, X \vdash A(b) \\
\frac{\Gamma, Y \vdash (\forall x) A(x) \quad \Gamma, X \vdash (\forall x) A(x)}{\Gamma, X \lor Y \vdash (\forall x) A(x)} \tag{\forall, r} \quad \tag{\forall, 1}
\]

is replaced by

\[
\Gamma, Y \vdash A(a) \quad \Gamma, X \vdash A(b) \\
\frac{\Gamma, Y \vdash (\forall x) A(x) \quad \Gamma, X \vdash (\forall x) A(x)}{\Gamma, X \lor Y \vdash (\forall x) A(x) \mid \Gamma, X \lor Y \vdash (\forall x) A(x)} \tag{\forall', r} \quad \tag{EC}
\]
Mid(hyper)sequent Theorem – Sketch of Proof

We replace all the applications of $(\forall, l)$ by applications of $(\forall', l)$.

\[
\frac{\Gamma, Y \vdash A(a)}{\Gamma, Y \vdash (\forall x)A(x)} \quad (\forall, r) \quad \frac{\Gamma, X \vdash A(b)}{\Gamma, X \vdash (\forall x)A(x)} \quad (\forall, r)
\]
\[
\frac{\Gamma, X \lor Y \vdash (\forall x)A(x)}{\Gamma, X \lor Y \vdash (\forall x)A(x)} \quad (\forall', 1)
\]
\[
\frac{\Gamma, X \lor Y \vdash (\forall x)A(x) \mid \Gamma, X \lor Y \vdash (\forall x)A(x)}{\Gamma, X \lor Y \vdash (\forall x)A(x)} \quad (EC)
\]

Can be transformed into

\[
\frac{\Gamma, Y \vdash A(a) \quad \Gamma, X \vdash A(b)}{\Gamma, X \lor Y \vdash A(a) \mid \Gamma, X \lor Y \vdash A(b)} \quad (\forall', 1)
\]
\[
\frac{\Gamma, X \lor Y \vdash (\forall x)A(x) \mid \Gamma, X \lor Y \vdash (\forall x)A(x)}{\Gamma, X \lor Y \vdash (\forall x)A(x)} \quad 2x(\forall, r)
\]
\[
\frac{\Gamma, X \lor Y \vdash (\forall x)A(x) \mid \Gamma, X \lor Y \vdash (\forall x)A(x)}{\Gamma, X \lor Y \vdash (\forall x)A(x)} \quad (EC)
\]

Tableaux’03 – p.31/63
FO Gödel Logic and the \((TT)\) rule

\[
\Gamma \vdash C \lor (A \rightarrow p) \lor (p \rightarrow B)
\]

\[
\frac{\Gamma \vdash C \lor (A \rightarrow B)}{}
\]

where \(p\) is does not occur in the conclusion. This rule, expressing the density of the ordered set of truth-values, was used by Takeuti and Titani to axiomatize first-order Gödel logic.
FO Gödel Logic and the $(TT)$ rule

\[
\frac{\Gamma \vdash C \lor (A \rightarrow p) \lor (p \rightarrow B)}{\Gamma \vdash C \lor (A \rightarrow B)}
\]

where \( p \) is does not occur in the conclusion. This rule, expressing the density of the ordered set of truth-values, was used by Takeuti and Titani to axiomatize first-order Gödel logic.

Takano (1984) posed the question whether a syntactical elimination of this rule is also possible. The hypersequent calculus for FO Gödel Logic allows one to give a positive answer to this question.
Quantified Propositional Gödel logic

Generalization of propositional Gödel logic obtained by adding quantifiers over propositional variables
Quantified Propositional Gödel logic

\[
v((\exists p)A) = \sup \{ v[w/p](A) : w \in [0, 1] \} \\
v((\forall p)A) = \inf \{ v[w/p](A) : w \in [0, 1] \}
\]
Quantified Propositional Gödel logic

\[ G_\infty = IL + (A \rightarrow B) \lor (B \rightarrow A) + \]

\[ Z[a] \rightarrow Y \quad (\exists \exists) \quad Y \rightarrow Z[a] \quad (\forall \forall) \]

where \( a \) does not occur in \( Y \).

\[ A[X] \rightarrow (\exists q)A[q] \quad ((\forall q)A[q]) \rightarrow A[X] \]

\( \lor - \text{Shift} : \quad ((\forall q)(A \lor B)) \rightarrow (A \lor (\forall q)B) \)

\( \text{Density} : \quad [(\forall q')(((A \rightarrow q') \lor (q' \rightarrow B))] \rightarrow (A \rightarrow B) \)

where \( q \) does not occur in \( A \) and \( q' \) occurs neither in \( A \) nor in \( B \).
M. Baaz and C. Fermüller and H. Veith "An Analytic Calculus for Quantified Propositional Gödel Logic". Proceedings of Tableaux 2000

Hypersequent Calculus for propositional Gödel Logic
Hypersequent Calculus for propositional Gödel Logic

\[
\frac{G \mid A[X], \Gamma \vdash B}{G \mid (\forall q)A[q], \Gamma \vdash B} (\forall, l)^0 \quad \frac{G \mid \Gamma \vdash A[a]}{G \mid \Gamma \vdash (\forall q)A[q]} (\forall, r)^0
\]

\[
\frac{G \mid A[a], \Gamma \vdash B}{G \mid (\exists q)A[q], \Gamma \vdash B} (\exists, l)^0 \quad \frac{G \mid \Gamma \vdash A[X]}{G \mid \Gamma \vdash (\exists q)A[q]} (\exists, r)^0
\]

\[
G \mid \Pi \vdash p \mid p, \Gamma \vdash C \quad (tt)
\]

\[
G \mid \Pi, \Gamma \vdash C
\]
Summary

1. Introduction to Hypersequents
2. Adding quantifiers
3. Advanced Topics
   - cut-elimination
   - automated generation of hypersequent calculi (Examples: basic fuzzy logics and global intuitionistic logic)
   - variants of the hypersequent framework (Examples: Łukasiewicz logic and Logics of bounded depth Kripke Models)
Cut-elimination in HC
Cut-elimination in HC

Gentzen Method
Proceeds by eliminating the uppermost cut by a double induction on the complexity of the cut formula and on the sum of its left and right ranks; where the right (left) rank of a cut is the number of consecutive (hyper)sequents containing the cut formula, counting upward from the right (left) upper sequent of the cut.
Cut-elimination in HC

If $G' | \Gamma \vdash A$ and $H' | \Gamma', A^{(n)} \vdash B$ are cut-free provable in a hypersequent calculus, so is $G' | H' | \Gamma, \Gamma' \vdash B$. 
Cut-elimination in HC

If $G' | \Gamma \vdash A$ and $H' | \Gamma' \vdash A^{(n)} \vdash B$ are cut-free provable in a hypersequent calculus, so is $G' | H' | \Gamma, \Gamma' \vdash B$.

Pb. with Gentzen’s method

\[
\begin{array}{c}
\Gamma, A \vdash B | \Gamma, A \vdash B \\
\hline
\Gamma, \Sigma \vdash A
\end{array}
\]

$(\text{EC})$

\[
\begin{array}{c}
\Gamma, A \vdash B \\
\hline
\Gamma, \Sigma \vdash B
\end{array}
\]

$(\text{cut})$
Cut-elimination in HC

If $G' \vdash A$ and $H' \vdash \Gamma', A^{(n)} \vdash B$ are cut-free provable in a hypersequent calculus, so is $G' \vdash H' \vdash \Gamma, \Gamma' \vdash B$.

**Pb. with Gentzen’s method**

\[
\frac{\Gamma, A \vdash B \mid \Gamma, A \vdash B}{\Gamma, A \vdash B} \quad (\text{EC})
\]
\[
\frac{\Gamma, A \vdash B \mid \Gamma, A \vdash B \quad \Sigma \vdash A}{\Sigma \vdash A} \quad (\text{cut})
\]
\[
\frac{\Gamma, A \vdash B \mid \Gamma, A \vdash B \quad \Sigma \vdash A}{\Gamma, \Sigma \vdash B \mid \Gamma, A \vdash B} \quad (\text{cut})
\]
\[
\frac{\Gamma, \Sigma \vdash B \mid \Gamma, A \vdash B \quad \Sigma \vdash A}{\Gamma, \Sigma \vdash B \mid \Gamma, \Sigma \vdash B} \quad (\text{cut})
\]
\[
\frac{\Gamma, \Sigma \vdash B \mid \Gamma, \Sigma \vdash B}{\Gamma, \Sigma \vdash B} \quad (\text{EC})
\]
Cut-elimination

Solution n. 1:
Use generalized cut rules
Cut-elimination

E.g. If $G \mid \Gamma_1 \vdash A \mid \ldots \mid \Gamma_n \vdash A$ and $H \mid \Sigma_1, A^{n_1} \vdash B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $H \mid G \mid \Gamma_1 \vdash \mid \ldots \mid \Gamma_n \vdash \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k$. 
Cut-elimination

E.g. If $G | \Gamma_1 \vdash A | \ldots | \Gamma_n \vdash A$ and $H | \Sigma_1, A^{n_1} \vdash B_1 | \ldots | \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $H | G | \Gamma_1 \vdash | \ldots | \Gamma_n \vdash | \Sigma_1 \vdash B_1 | \ldots | \Sigma_k \vdash B_k$. It holds in Classical Logic.
**Cut-elimination**

If \( G := G | \Gamma_1 \vdash A | \ldots | \Gamma_n \vdash A \) and \( H := H | \Sigma_1, A^{n_1} \vdash B_1 | \ldots | \Sigma_k, A^{n_k} \vdash B_k \) are cut-free provable, so is \( H | \Gamma_1 \vdash \ldots | \Gamma_n \vdash \Sigma_1 \vdash B_1 | \ldots | \Sigma_k \vdash B_k \).

\[
\begin{array}{c}
\dfrac{H | \Sigma_k, A^{n_k} \vdash B_k}{H} \quad \text{(EC)}
\end{array}
\]

\[
\begin{array}{c}
\dfrac{H | \Sigma_k, A^{n_k} \vdash B_k \quad G}{H \quad G} \quad \text{(cut)}
\end{array}
\]

\[
\begin{array}{c}
\dfrac{H | \Sigma_1 \vdash B_1 \quad \ldots \quad \Sigma_k \vdash B_k \quad \Sigma_k \vdash B_k \quad \Gamma_1 \vdash \ldots \quad \Gamma_n \vdash}{H \quad G \quad \Sigma_1 \vdash B_1 \quad \ldots \quad \Sigma_k \vdash B_k \quad \Sigma_k \vdash B_k} \quad \text{(cut)}
\end{array}
\]

\[
\begin{array}{c}
\dfrac{H \quad G \quad \Sigma_1 \vdash B_1 \quad \ldots \quad \Sigma_k \vdash B_k \quad \Sigma_k \vdash B_k \quad \Gamma_1 \vdash \ldots \quad \Gamma_n \vdash}{H \quad G \quad \Sigma_1 \vdash B_1 \quad \ldots \quad \Sigma_k \vdash B_k \quad \Sigma_k \vdash B_k} \quad \text{(EC)}
\end{array}
\]
Cut-elimination

For Classical Logic
If $G \vdash \Gamma_1 \vdash A \vdots \vdash \Gamma_n \vdash A$ and $H \vdash \Sigma_1, A^{n_1} \vdash B_1 \vdots \vdash \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $G \vdash H \vdash \Gamma_1 \vdash A \vdots \vdash \Gamma_n \vdash A \vdash \Sigma_1 \vdash B_1 \vdots \vdash \Sigma_k \vdash B_k$.

For Gödel Logic
If $G \vdash \Gamma_1 \vdash A \vdots \vdash \Gamma_n \vdash A$ and $H \vdash \Sigma_1, A^{n_1} \vdash B_1 \vdots \vdash \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $G \vdash H \vdash \Gamma, \Sigma_1 \vdash B_1 \vdots \vdash \Gamma, \Sigma_k \vdash B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$. 
Cut-elimination

Cut-elimination


Schütte-Tait Method
Proceeds by eliminating the largest cut (w.r.t. the number of connectives and quantifiers).
Cut-elimination


Schütte-Tait Method

Proceeds by eliminating the largest cut (w.r.t. the number of connectives and quantifiers).

A cut is not shifted upward but simply reduced (replaced by smaller cuts) in the place in which the cut-formula is introduced.
Cut-elimination

Cut-elimination

\[
\begin{array}{c}
\vdash d & \vdash d' \\
G \mid \Gamma' \vdash X & H \mid \Gamma, X \vdash B \\
\hline
G \mid H \mid \Gamma, \Gamma' \vdash B
\end{array}
\]

(cut)

Two cases:
1. one first inverts (reduces the complexity of the cut formula in) one of the two sides of the cut; (Inversion Lemma)
2. the cut is then reduced (replaced by smaller cuts) in the place in which the cut formula is introduced; (Reduction Lemma)
Cut-elimination

\[
\vdots d \\
G \mid \Gamma' \vdash X \\
\vdots d'
\]

\[
H \mid \Gamma, X \vdash B
\]

\[
G \mid H \mid \Gamma, \Gamma' \vdash B
\]

\text{(cut)}

Two cases:

- atomic cut
- non atomic cut
Cut-elimination

\[
\frac{\vdash d}{G \mid \Gamma' \vdash X^*} \quad \frac{\vdash d'}{H \mid \Gamma, X^* \vdash B}
\]

\[
G \mid H \mid \Gamma, \Gamma' \vdash B \quad \text{(cut)}
\]

Two cases:

- atomic cut
- non atomic cut
Cut-elimination

\[
\begin{array}{c}
\vdash d \\
G \mid \Gamma' \vdash X^* \\
H \mid \Gamma, X^* \vdash B \\
\hline
G \mid H \mid \Gamma, \Gamma' \vdash B \\
\end{array}
\]

\text{(cut)}

Two cases:

- atomic cut
  Replace \( X^* \) in \( d' \) by \( \Gamma' \).
  Two possibilities:
    - \( X^* \) originates in a weakening rule.
    - \( X^* \) originates in an axiom \( X^* \vdash X \).

We get a proof of \( G \mid H \mid \Gamma, \Gamma' \vdash B \) starting from a proof of \( G \mid \Gamma' \vdash X \)

- non atomic cut

Tableaux'03 – p.39/63
Cut-elimination

\[ \begin{array}{c}
\vdash d \\
G \mid \Gamma' \vdash X^* \\
H \mid \Gamma, X^* \vdash B \\
\hline \\
G \mid H \mid \Gamma, \Gamma' \vdash B \\
\end{array} \] (cut)

Two cases:

- atomic cut
- non atomic cut

1. one first inverts (reduces the complexity of the cut formula in) one of the two sides of the cut; (Inversion Lemma)

2. the cut is then reduced (replaced by smaller cuts) in the place in which the cut formula is introduced; (Reduction Lemma).
Cut-elimination

Inversion Lemma

1. If \( d \) is a derivation of \( G_j \rightarrow A \land B \rightarrow C \) one can find \( d_1 \); \( G_j \rightarrow C \) and \( d_2 \); \( G_j \rightarrow B \rightarrow C \).

2. If \( d; G_j \rightarrow A \rightarrow B \) then one can find \( d_1 ; G_j \rightarrow A \) and \( d_2 ; G_j \rightarrow B \rightarrow C \).

3. If \( d; G_j \rightarrow A \rightarrow ! B \) one can find \( d_1 ; G_j \rightarrow B \rightarrow A \).

4. If \( d; G_j \rightarrow A \rightarrow (x) \rightarrow C \) one can find \( d_1 ; G_j \rightarrow A \rightarrow (a) \rightarrow C \) such that \( c(d_i) \rightarrow C \) and the \( l(d_i) \rightarrow C \), \( i = 1, 2 \).
Cut-elimination

Inversion Lemma
(reduces the complexity of the cut formula on the “invertible” side)

1. If \( d \) is a derivation of \( \Gamma \) one can find \( d_1 \) and \( d_2 \).

2. If \( d \) is a derivation of \( \Gamma \) then one can find \( d_1 \) and \( d_2 \).

3. If \( d \) is a derivation of \( \Gamma \) then one can find \( d_1 \) and \( d_2 \).

4. If \( d \) is a derivation of \( \Gamma \) then one can find \( d_1 \) and \( d_2 \).

5. If \( d \) is a derivation of \( \Gamma \) then one can find \( d_1 \) and \( d_2 \).
Cut-elimination

Inversion Lemma

1. If $d$ is a derivation of $G \mid \Gamma, A \lor B \vdash C$ one can find $d_1$, $G \mid \Gamma, A \vdash C$ and $d_2$, $G \mid \Gamma, B \vdash C$

2. If $d$, $G \mid \Gamma \vdash A \land B$ then one can find $d_1$, $G \mid \Gamma \vdash A$ and $d_2$, $G \mid \Gamma \vdash B$

3. If $d, G \mid \Gamma \vdash A \rightarrow B$ one can find $d_1$, $G \mid \Gamma, A \vdash B$

4. If $d, G \mid \Gamma, \exists x A(x) \vdash C$ one can find $d_1$, $G \mid \Gamma, A(a) \vdash C$

5. If $d, G \mid \Gamma \vdash \forall x A(x)$ one can find $d_1$, $G \mid \Gamma \vdash A(a)$

such that $c(d_i) \leq c(d)$ and the $l(d_i) \leq l(d)$, $i = 1, 2$. 
Cut-elimination

Reduction Lemma

*E.g.*:

\[ \vdots \quad G' | \Psi \vdash B \quad G' | \Psi, D \vdash C' \quad (\rightarrow,1) \\]

\[ \vdots \quad G | \Sigma \vdash B \rightarrow D \quad H | \Gamma, B \rightarrow D \vdash C \quad (\text{cut}) \\]

Using \( G | \Sigma, B \vdash D \) (Inversion Lemma)
Cut-elimination

Reduction Lemma

E.g.:

\[
\begin{align*}
G' | \Psi & \vdash B & G' | \Psi, D & \vdash C' \\
\downarrow & & \downarrow & & \downarrow \\
G | \Sigma & \vdash B \rightarrow D & H | \Gamma, (B \rightarrow D)^* & \vdash C \\
\downarrow & & \downarrow & & \downarrow \\
G | H & \vdash \Gamma, \Sigma & \vdash C \\
\end{align*}
\]

Using \( G | \Sigma, B \vdash D \) (Inversion Lemma)
Automated Generation of HC

Logic $L$

cut free (hyper)sequent calculus
Automated Generation of HC

Logic $L$
(cut free (hyper)sequent calculus)

+ 

”certain” properties (★)
Automated Generation of HC

Logic $L$
(cut free (hyper)sequent calculus)

+ 

”certain” properties (⋆)

⇒

Logic $L^*$
(cut free hypersequent sequent calculus)
Automated Generation of HC

Properties: (We consider logics as characterized by their Hilbert style systems)

1. \((A \rightarrow B) \lor (B \rightarrow A)\)

2. modality \(\Delta\) s.t.

\[
\begin{align*}
(1) \ & \Delta A \rightarrow A \ \\
(2) \ & \Delta A \rightarrow \Delta \Delta A \ \\
(3) \ & \Delta A \lor \neg \Delta A \ \\
(4) \ & \Delta (A \lor B) \rightarrow \Delta A \lor \Delta B \ \\
(5) \ & \Delta (A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)
\end{align*}
\]

\[
\frac{A}{\Delta A} \quad (\Delta \ rule)
\]

(related to the modality in S4)
Automated Generation of HC

The involved logics

- should admit some suitable rules
- their cut-free (hyper)sequent calculi have to have permulation rules, ”standard” rules for connectives...
Automated Generation of HC

(M. Baaz, *. ”Generating Inference Systems for logics with linearity”. In preparation)

Logic $L$

((single conclusioned) cut free (hyper)sequent calculus $S$)
Automated Generation of HC

Logic $L$

((single conclusioned) cut free (hyper)sequent calculus $S_n$)

$+$

$(A \rightarrow B) \lor (B \rightarrow A)$
Automated Generation of HC Logic $L$
((single conclusioned) cut free (hyper)sequent calculus $S$)

\[(A \rightarrow B) \lor (B \rightarrow A)\]  
\[\implies \text{Logic } L^*\]
Automated Generation of HC

Logic $L$
((single conclusioned) cut free (hyper)sequent calculus $S$)

$+$

$(A \rightarrow B) \vee (B \rightarrow A)$

$\implies$ Logic $L^*$

cut free hypersequent sequent calculus:
Automated Generation of HC

Logic $L$

((single conclusioned) cut free (hyper)sequent calculus $S^+$)

$+(A \rightarrow B) \lor (B \rightarrow A)$

$\implies$ Logic $L^*$

(cut free hypersequent sequent calculus):

- hypersequent version of $S$

- $\frac{G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma'_1 \vdash A'}{G|G'|\Gamma, \Gamma'_1 \vdash A|\Gamma', \Gamma_1 \vdash A'}$ (com)
Example: Basic fuzzy logics

- MTL
- Urquhart’s C logic (versions I and II)
Example: Basic fuzzy logics

- MTL
- Urquhart’s C logic (versions I and II)

Main formalization of fuzzy logic: Gödel, Łukasiewicz and Product logic.

Example: Basic fuzzy logics

- MTL
- Urquhart’s C logic (versions I and II)

Main formalization of fuzzy logic: Gödel, Łukasiewicz and Product logic. Correspond to the most important t-norms that are the main tool to combine fuzzy information.

By a $t$-norm one means some binary operation $\ast$ in the real unit interval $[0, 1]$ which is associative, commutative, non-decreasing in both arguments and which has 1 as a neutral element.
Example: Basic fuzzy logics

- MTL
- Urquhart’s C logic (versions I and II)

Main formalization of fuzzy logic: Gödel
\[ x \ast y = \min(x, y) , \text{Łukasiewicz} \]
\[ x \ast y = \max(0, x + y - 1) \text{ and Product logic} \]
\[ x \ast y = x \cdot y . \]

Correspond to the most important t-norms that are the main tool to combine fuzzy information.

By a \( t \)-norm one means some binary operation \( \ast \) in the real unit interval \([0, 1]\) which is associative, commutative, non-decreasing in both arguments and which has 1 as a neutral element.
Basic fuzzy logics: MTL

As well as defining logics based on particular t-norm, logics may also be based on classes of t-norms. (Logics are identified with the formulas valid in all the logics based on t-norms from that class)
Basic fuzzy logics: MTL

As well as defining logics based on particular t-norm, logics may also be based on classes of t-norms. (Logics are identified with the formulas valid in all the logics based on t-norms from that class) (F. Esteva and L. Godo. ”Monoidal t-norm based Logic: towards a logic for left-continuous t-norms”. Fuzzy Sets and Systems. 2001”)

MTL is the logic of *left-continuous t-norms*. 

MTL = aMAILL + (A \! \! B) _ (B \! \! A)
Basic fuzzy logics: MTL

As well as defining logics based on particular t-norm, logics may also be based on classes of t-norms. (Logics are identified with the formulas valid in all the logics based on t-norms from that class)

MTL is the logic of *left-continuous t-norms*.

\[
\text{MTL} = \text{aMAILL} + (A \to B) \lor (B \to A)
\]

Tableaux’03 – p.46/63
Basic fuzzy logics: C

Basic fuzzy logics: C

Ax1. $A \rightarrow (B \rightarrow A)$

Ax2. $(A \rightarrow B) \rightarrow [(C \rightarrow A) \rightarrow (C \rightarrow B)]$

Ax3. $[A \rightarrow (C \rightarrow B)] \rightarrow [C \rightarrow (A \rightarrow B)]$

Ax4. $(A \land B) \rightarrow A$

Ax5. $(A \land B) \rightarrow B$

Ax6. $A \rightarrow [C \rightarrow (A \land C')]$

Ax7. $A \rightarrow (A \lor B)$

Ax8. $B \rightarrow (A \lor B)$

Ax9. $[(A \rightarrow C') \land (B \rightarrow C')] \rightarrow [(A \lor B) \rightarrow C]$

Lin. $(A \rightarrow B) \lor (B \rightarrow A)$
Basic fuzzy logics: C

Urquhart claimed that C is semantically characterized by model structures on ordered commutative monoids.
Basic fuzzy logics: C

To obtain completeness in (A. Urquhart. "Many-Valued Logic". In: Handbook of Philosophical Logic, Vol. III. Second Edition. 2001) Urquhart added to C the following axioms:

**U1.** \(((A \rightarrow B) \land (A \rightarrow C')) \rightarrow (A \rightarrow (B \land C'))\)

**U2.** \(((A^k \rightarrow C') \land (B^k \rightarrow C')) \rightarrow ((A \lor B)^k \rightarrow C')\)

for every \(k \geq 2\), where \(A^k \rightarrow C\) stands for \(A \rightarrow (A \rightarrow \ldots \ldots (A \rightarrow C) \ldots ))\).
Basic fuzzy logics
Basic fuzzy logics

Hilbert system for IL without contraction
Basic fuzzy logics

Hilbert system for IL **without contraction** with the following axioms for AND

- \((A \land B) \rightarrow A\) \((A \land B) \rightarrow B\),
- \(((A \rightarrow B) \land (A \rightarrow C)) \rightarrow (A \rightarrow (B \land C'))\),
- \((A \rightarrow (B \rightarrow C')) \rightarrow ((A \odot B) \rightarrow C')\),
- \(A \rightarrow [C' \rightarrow (A \odot C')]

- \((A \land B) \rightarrow A\), \((A \land B) \rightarrow B\),
- \(A \rightarrow [C' \rightarrow (A \land C')]

Tableaux’03 – p.49/63
Basic fuzzy logics

Hilbert system for IL without contraction with the following axioms for AND 
\[ + (A \to B) \lor (B \to A) \]

- \[(A \land B) \to A \quad (A \land B) \to B, \]
- \[((A \to B) \land (A \to C)) \to (A \to (B \land C')) \],
- \[(A \to (B \to C')) \to ((A \circ B) \to C'), \]
- \[A \to [C \to (A \circ C')] : MTL \]

- \[(A \land B) \to A, (A \land B) \to B, \]
- \[A \to [C \to (A \land C')] : \text{Urquart’s C logic (version I)} \]
HC for basic fuzzy logics

LJ **without contraction**

(Ono, H. and Komori, Y. ”Logics without the contraction rule ”. J. of Symbolic Logic. 1985)
HC for basic fuzzy logics

\textbf{LJ without contraction}

(Ono, H. and Komori, Y. ”Logics without the contraction rule”. J. of Symbolic Logic. 1985)

\[
(\land, r)_m \quad \frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \land B} \quad (\land, l)_m \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \land B \vdash C}
\]

\[
(\land, r)_a \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \land B} \quad (\land_i, l)_a \quad \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \land A_2 \vdash C}
\]
HC for Basic fuzzy logics

LJ without contraction

- both additive and multiplicative rules for AND (i.e. $\land$ and $\odot$)

- $(\land, r)_m$ \[ \frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \land B} \]

- $(\land_i, l)_a$ \[ \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \land A_2 \vdash C} \]

- additive rules for AND
HC for Basic fuzzy logics

LJ without contraction

- both additive and multiplicative rules for AND (i.e. $\land$ and $\otimes$) (calculus for aMAILL)

\[
\begin{align*}
\Gamma \vdash A & \quad \Gamma' \vdash B \\
\Gamma, \Gamma' \vdash A \land B & \\
\Gamma, A_i \vdash C & \\
\Gamma, A_1 \land A_2 \vdash C
\end{align*}
\]

(calculus for $C = (A \rightarrow B) \lor (B \rightarrow A)$)

- additive rules for AND
HC for Basic fuzzy logics

Hypersequent version of LJ without contraction +

\[
\frac{G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma'_1 \vdash A'}{G|G'|\Gamma, \Gamma'_1 \vdash A|\Gamma', \Gamma_1 \vdash A'} \quad (com)
\]

- MTL: both additive and multiplicative rules for AND (i.e. \( \land \) and \( \odot \))

- Urquart’s C logic I:

\[
\frac{G|\Gamma \vdash A \quad G'|\Gamma' \vdash B}{G|G'|\Gamma, \Gamma' \vdash A \land B}
\]

\[
\frac{G|\Gamma, A_i \vdash C}{G|\Gamma, A_1 \land A_2 \vdash C'}
\]

- Urquart’s C logic II: additive rules for AND
Urquhart defined the new C logic by adding to C the following axioms:

**U1.** $((A \rightarrow B) \land (A \rightarrow C')) \rightarrow (A \rightarrow (B \land C'))$

**U2.** $((A^k \rightarrow C) \land (B^k \rightarrow C')) \rightarrow ((A \lor B)^k \rightarrow C')$

for every $k \geq 2$, where $A^k \rightarrow C$ stands for $\underbrace{A \rightarrow (A \rightarrow (\ldots (A \rightarrow C) \ldots ))}_{k \text{ times}}$. 
Urquhart defined the new C logic by adding to C the following axioms:

**U1.** \(((A \rightarrow B) \land (A \rightarrow C')) \rightarrow (A \rightarrow (B \land C'))\)

**U2.** \(((A^k \rightarrow C) \land (B^k \rightarrow C')) \rightarrow ((A \lor B)^k \rightarrow C')\)

**Fact 1:** the hypersequent version of LJ without contraction with the additive rules for AND characterize the logic \(C + U1\).
Urquhart defined the new C logic by adding to C the following axioms:

**U1.** \(((A \rightarrow B) \land (A \rightarrow C')) \rightarrow (A \rightarrow (B \land C'))\)

**U2.** \(((A^k \rightarrow C') \land (B^k \rightarrow C')) \rightarrow ((A \lor B)^k \rightarrow C')\)

**Fact 1:** the hypersequent version of LJ without contraction with the additive rules for AND characterize the logic \(C + U1\).

**Fact 2:** Axiom U2 is derivable in this calculus (*, ”On Urquhart’s C logic”. Proc. of ISMVL’2000)
Urquhart defined the new C logic by adding to C the following axioms:

**U1.** \(((A \rightarrow B) \land (A \rightarrow C')) \rightarrow (A \rightarrow (B \land C'))\)

**U2.** \(((A^k \rightarrow C') \land (B^k \rightarrow C')) \rightarrow ((A \lor B)^k \rightarrow C')\)

**Fact 1:** the hypersequent version of LJ without contraction with the additive rules for AND characterize the logic \(C + U1\).

**Fact 2:** Axiom U2 is derivable in this calculus

**Corollary**

- The above hypersequent calculus characterize the logic C (version II)
- Axiom U2 is redundant in the axiomatization of C (version II)
Automated Generation of HC with $\Delta$

Logic $L$

((single conclusioned) cut free sequent calculus $S'$)
Automated Generation of HC with $\Delta$

Logic $L$

((single conclusioned) cut free sequent calculus $S'$)

+ the modality $\Delta$

(1) $\Delta A \rightarrow A$

(2) $\Delta A \rightarrow \Delta \Delta A$

(3) $\Delta A \lor \neg \Delta A$

(4) $\Delta (A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)$

\[
\frac{A}{\Delta A} \quad (\Delta \text{ rule})
\]

(if $L$ is a first order logic):

(5) $\forall x \Delta A(x) \rightarrow \Delta \forall x A(x))$
Automated Generation of HC with $\Delta$

Logic $L$

$\text{((single conclusioned) cut free sequent calculus } S')$

$+ \text{ the modality } \Delta$

$\implies \text{Logic } L^\Delta$

Tableaux’03 – p.53/63
Automated Generation of HC with $\Delta$

Logic $L$

((single conclusioned) cut free sequent calculus $S$)

+ the modality $\Delta$

$\implies$ Logic $L^\Delta$

(cut free hypersequent sequent calculus):
Automated Generation of HC with $\Delta$

Logic $L$
((single conclusioned) cut free sequent calculus $S'$) + the modality $\Delta$

$\implies$ Logic $L^\Delta$
(cut free hypersequent sequent calculus):
- hypersequent version of $S$

\[
\frac{G \mid \Gamma, A \vdash C}{G \mid \Gamma, \Delta A \vdash C} (\Delta, l) \quad \frac{G \mid \Delta \Gamma \vdash A}{G \mid \Delta \Gamma \vdash \Delta A} (\Delta, r) \quad \frac{G \mid \Delta \Gamma, \Gamma' \vdash A}{G \mid \Delta \Gamma \vdash \Gamma' \vdash A} (cl_{\Delta}, l)
\]
Ex: Global Int (Fuzzy) Logic

• Global Intuitionistic Logic = FO Intuitionistic Logic + Δ

• Global Intuitionistic Fuzzy Logic = FO Gödel Logic + Δ
Global Intuitionistic Logic = FO Intuitionistic Logic + $\Delta$

Sequent calculus: obtained by adding to (a suitable modification of) Maehara’s $LJ'$ the rules for $\square$ of the sequent calculus for the modal logic $S5$, i.e.

\[
\frac{\Gamma, A \vdash \Sigma}{\Gamma, \Delta A \vdash \Sigma} (\square_l)' \quad \frac{\Gamma \vdash A, \Sigma}{\Gamma \vdash \Delta A, \Sigma} (\square_r)'
\]

(where $\overline{\Sigma}$ and $\overline{\Gamma}$ are sets of $\Delta$-closed formulas)
Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = FO Intuitionistic Logic + △
Sequent calculus: obtained by adding to (a suitable modification of) Maehara’s $LJ'$ the rules for □ of the sequent calculus for the modal logic $S5$, $LJ'$ is an equivalent version of Gentzen’s $LJ$ where the restriction to at most one formula in the succedent of sequents applies not generally but only in the case of the right rules for $→, \neg$ and $\forall$. E.g., the $→$ right rule is

$$\Gamma, A \vdash B, (\Sigma) \quad (\rightarrow, r)'$$
Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = FO Intuitionistic Logic + △

Sequent calculus: obtained by adding to (a suitable modification of) Maehara’s $LJ'$ the rules for $\square$ of the sequent calculus for the modal logic $S5$.

This calculus does not admit elimination of cuts. (E.g. the sequent

$$\vdash \forall x \neg \square A(x), \exists x A(x)$$

is derivable in this calculus only using cuts.)
Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta
Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
  Cut free Hypersequent Calculus = Hypersequent version of LJ with in addition

\[
\frac{G | \Gamma, A \vdash C}{G | \Gamma, \Delta A \vdash C} \quad (\Delta, l) \quad \frac{G | \Delta \Gamma \vdash A}{G | \Delta \Gamma \vdash \Delta A} \quad (\Delta, r)
\]

\[
\frac{G | \Delta \Gamma, \Gamma' \vdash A}{G | \Delta \Gamma \vdash | \Gamma' \vdash A} \quad (cl_\Delta, l)
\]

- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta
  Cut free Hypersequent Calculus = Hypersequent version of LJ with in addition \((\Delta, l), (\Delta, r), (cl_\Delta, l)\) and (com).
Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta

In both cases, the hypersequent calculus allows one to prove a (suitable version of the) midsequent theorem.
Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta

In both cases, the hypersequent calculus allows one to prove a (suitable version of the) midsequent theorem.

(For GI it holds for proofs in which \((\vee, l)\) is applied only to formulas where (at least) one of the disjuncts is a \(\Delta\)-formula (i.e. \(\Delta A \vee B\))

*, ”A proof-theoretical investigation of global intuitionistic (fuzzy) logic”. Draft. 2003

Tableaux’03 – p.55/63
Variants of the Hypersequent Framework

E.g., IL + A_ A (Classical logic)

A ! B _ (Goedel logic)

A :: A (LQ)

Tableaux’03 – p.56/63
Variants of the Hypersequent Framework

"simple" disjunctive conditions
Variants of the Hypersequent Framework

E.g.,

\[ IL + \]

\[ \neg A \lor A \quad \text{(Classical logic)} \]

\[ (A \rightarrow B) \lor (B \rightarrow A) \quad \text{(Goedel logic)} \]

\[ \neg A \lor \neg \neg A \quad \text{(LQ)} \]
Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models

- Łukasiewicz Logic

 Łukasiewicz Logic: \( a \text{MALL} + ( (A \lor B) \lor B \lor A ) \lor A \lor B \) allows one to define the additive connectives \(^\lor\) and \(_\lor\) over the multiplicative ones \(\land\) and \(\lor\). (E.g. \(A \lor B = A \land (A \lor B) \lor B \lor A \lor B\))
Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models: IL + the axiom scheme \((Bd_k)\) recursively defined as follows:

\[
\begin{align*}
(Bd_1) & \quad A_1 \lor \neg A_1 \\
(Bd_{i+1}) & \quad A_{i+1} \lor (A_{i+1} \rightarrow (Bd_i))
\end{align*}
\]

- Łukasiewicz Logic: aMALL +

\[
((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)
\]
Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models: IL + the axiom scheme \((Bd_k)\) recursively defined as follows:

\[
(Bd_1) \quad A_1 \lor \neg A_1
\]
\[
(Bd_{i+1}) \quad A_{i+1} \lor (A_{i+1} \rightarrow (Bd_i))
\]

- Łukasiewicz Logic: aMALL +

\[
((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)
\]

\[
A \lor B = \min\{A, B\} \quad A \land B = \max\{A, B\}
\]
\[
A \oplus B = \min\{1, A + B\} \quad A \odot B = \max\{0, A + B\}
\]
\[
A \rightarrow B = \min\{1, A + B - 1\} \quad \neg A = 1 - A
\]
Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models: IL + the axiom scheme \((Bd_k)\) recursively defined as follows:

\[
\begin{align*}
(Bd_1) & \quad A_1 \lor \neg A_1 \\
(Bd_{i+1}) & \quad A_{i+1} \lor (A_{i+1} \rightarrow (Bd_i))
\end{align*}
\]

- Łukasiewicz Logic: aMALL +

\[
((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)
\]

allows one to define the additive connectives \(\land\) and \(\lor\) over the multiplicative ones \(\odot\) and \(\oplus\). (E.g. \(A \land B = A \odot (A \rightarrow B)\))
HC for Logics of Kripke models with depth $\leq k$

**Idea:** introduce additional structure on hypersequents
HC for Logics of Kripke models with depth $\leq k$

**Idea**: introduce additional structure on hypersequents

Elimination of the (EE) rule.

(a hypersequent in this framework is a suitable ordered sequence of components)
HC for Logics of Kripke models with depth $\leq k$

Idea: introduce additional structure on hypersequents

Elimination of the (EE) rule.
(a hypersequent in this framework is a suitable ordered sequence of components)

Analytic calculus for these logics = (Suitable) "hypersequent" calculus for IL + a single uniform rule (for each $k$).
HC for Logics of Kripke models with depth $\leq k$

Idea: introduce additional structure on hypersequents

Elimination of the (EE) rule.
(a hypersequent in this framework is a suitable ordered sequence of components)
Analytic calculus for these logics = (Suitable) ”hypersequent” calculus for IL + a single uniform rule (for each $k$).

HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculi are defined by dualizing hypersequent calculi: Given a hypersequent $\Gamma_1 \vdash \Delta_1 | \ldots | \Gamma_n \vdash \Delta_n$, each component $\Gamma_i \vdash \Delta_i$ translates into the set of signed formulas $T(\Gamma_i) \cup F(\Delta_i)$ where $T(\Gamma_i) = \{ TA \mid A \in \Gamma_i \}$ and $F(\Delta_i) = \{ FA \mid A \in \Delta_i \}$. Thus the above hypersequent translates into the $h$-set

$$T(\Gamma_1) \cup F(\Delta_1) | \ldots | T(\Gamma_n) \cup F(\Delta_n).$$
HC for Logics of Kripke models with depth $\leq k$

By dualizing hypersequent calculi one can define tableau calculi as follows: Given a hypersequent $\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$, each component $\Gamma_i \vdash \Delta_i$ translates into the set of signed formulas $T(\Gamma_i) \cup F(\Delta_i)$ where $T(\Gamma_i) = \{ TA \mid A \in \Gamma_i \}$ and $F(\Delta_i) = \{ FA \mid A \in \Delta_i \}$. Thus the above hypersequent translates into the $h$-set

$$T(\Gamma_1) \cup F(\Delta_1) \mid \ldots \mid T(\Gamma_n) \cup F(\Delta_n).$$

We say that an $h$-set $S_1 \mid \ldots \mid S_n$ is realized in $K$, if all the sets $S_j$, with $j = 1, \ldots, n$, are realized in $K$ (Given a Kripke model $K = \langle P, \leq, v \rangle$ and a signed formula $H$, we say that $\alpha \in P$ realizes $H$ if $H \equiv TX$ and $\alpha \models X$, or $H \equiv FX$ and $\alpha \models \neg X$.)
HC for Logics of Kripke models with depth \( \leq k \)

**Ex.** (Hyper)Tableau calculus for IL

**External Structural Rules**

\[
\frac{\Psi | \Phi}{\Psi} \quad \frac{\Psi | S}{\Psi | S} \quad \frac{\Psi | S_1 | S_2 | \Phi}{\Psi | S_2 | S_1 | \Phi}
\]

**Logical Rules**

\[
\frac{\Psi | S, \text{TA}_i}{\Psi | S, \text{TA}_i} \quad \text{T}_{\land_i} \quad \text{for } i = 1, 2
\]

\[
\frac{\Psi | S, \text{TF} (A \rightarrow B)}{\Psi | S, \text{TF} (A \rightarrow B)} \quad \text{T}_{\rightarrow} \quad \text{for } i = 1, 2
\]

\[
\frac{\Psi | S, \text{TF} (A \lor B)}{\Psi | S, \text{TF} (A \lor B)} \quad \text{T}_{\lor} \quad \text{for } i = 1, 2
\]

\[
\frac{\Psi | S, \text{TF} (A \land B)}{\Psi | S, \text{TF} (A \land B)} \quad \text{F}_{\land}
\]

\[
\frac{\Psi | S, \text{TF} (A \lor B)}{\Psi | S, \text{TF} (A \lor B)} \quad \text{F}_{\lor_i} \quad \text{for } i = 1, 2
\]

\[
\frac{\Psi | S, \text{TF} (A \land B)}{\Psi | S, \text{TF} (A \land B)} \quad \text{F}_{\land}
\]

\[
\frac{\Psi | S, \text{TF} (A \lor B)}{\Psi | S, \text{TF} (A \lor B)} \quad \text{F}_{\lor_i} \quad \text{for } i = 1, 2
\]

\[
\frac{\Psi | S, \text{TF} (A \rightarrow B)}{\Psi | S, \text{TF} (A \rightarrow B)} \quad \text{T}_{\rightarrow}
\]

\[
\frac{\Psi | S, \text{TF} (A \rightarrow B)}{\Psi | S, \text{TF} (A \rightarrow B)} \quad \text{F}_{\rightarrow}
\]

\[
S^{T} = \{TX | TX \in S\}
\]
New interpretation: We say that an h-set $S_1 | \ldots | S_n$ is path-realized in a Kripke model $K$, if there exists a path $\alpha = \alpha_1, \ldots, \alpha_n \in K$ ($\alpha_1 \leq \ldots \leq \alpha_n$) such that $\alpha_i$ realizes $S_i$, for every $1 \leq i \leq n$. 
HC for Logics of Kripke models with depth $\leq k$

We say that an h-set $S_1 | \ldots | S_n$ is path-realized in a Kripke model $K$, if there exists a path $\alpha = \alpha_1, \ldots, \alpha_n \in K (\alpha_1 \leq \ldots \leq \alpha_n)$ such that $\alpha_i \triangleright S_i$, for every $1 \leq i \leq n$.

In the new interpretation of h-sets the external exchange rule does not hold.
HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculus for IL became External Structural Rules:

\[
\begin{align*}
\Psi & \mid \Phi \quad \text{EC}_r \\
\Psi & \quad \text{EC}_1 \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi \\
\Psi & \mid \Phi
\end{align*}
\]
(Hyper)Tableau calculus for IL became Logical Rules
E.g.

\[ \Psi \mid S, \top(A_1 \land A_2) \mid \Psi' \]

\[ \frac{\Psi \mid S, \top A_i \mid \Psi'}{\Psi \mid S, \top \wedge_i \mid \Psi'} \]
HC for Logics of Kripke models with depth \( \leq k \)

(Hyper)Tableau calculus for IL became Logical Rules

But

\[
\begin{array}{c}
\Psi \mid S, F(A \rightarrow B) \mid \Psi' \\
\hline
\Psi \mid S^T, TA, FB
\end{array}
\]

Tableaux’03 – p.60/63
HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculus for IL became Logical Rules

\[ \Psi | S_0 | \ldots | S_k | \Psi' \] one derives

\[ \Psi | S_0, S_1 | \Psi' \parallel \Psi | S_0 | S_1, S_2 | \Psi' \parallel \ldots \parallel \Psi | S_0 | \ldots | S_{k-2} | S_{k-1}, S_k | \Psi'. \]
HC for Łukasiewicz Logic

\[
\begin{align*}
&\neg (A \land B) \land \neg B \\
&\neg (B \land A) \\
&\neg (\neg A) \\
&\neg (\neg B)
\end{align*}
\]


Note: the interpretation of components (i.e. sequents) is changed! `\(\land\)` is not interpreted as \(\land\) \(A \land B\) \(\lor\) \(B \land A\) but using the characteristic model \([1; 0]\) of Łukasiewicz Logic. I.e. if for all valuations \(v\) for \([1; 0]\), \(A \land B \land \neg B\) \(\land\) \(B \land A\) \(\land\) \(\neg A\) \(\land\) \(\neg B\) (where \(A \land B \land \neg B\) \(\land\) \(B \land A\) \(\land\) \(\neg A\) \(\land\) \(\neg B\)).
HC for Łukasiewicz Logic

- Infinite-valued Łukasiewicz logic is one of the main formalizations of Fuzzy Logic (Hajék 1998)
- Formulae in Łukasiewicz logic stand to particular geometric functions as formulae in classical logic stand to boolean functions. (McNaughton 1951)
- Łukasiewicz Logic(s) formalise Ulam’s game (a variant of the game of Twenty Questions where errors/lies are allowed in the answers) (Mundici 1992)
HC for Łukasiewicz Logic

D. Mundici, N. Olivetti. ”Resolution and model building in the infinite-valued calculus of Łukasiewicz”. TCS. 1998. (intersecting hyperplanes)
S. Aguzzoli, *. ”Finiteness in infinite-valued Łukasiewicz logic”. J. of Logic Language and Information. 2000. (reductions to finite-valued Łukasiewicz Logics)
HC for Łukasiewicz Logic

3-valued Łukasiewicz Logic

\[\text{aMALL } + \neg A \lor A \oplus A\]

hypersequent calculus for aMALL +

\[
G|G'| \Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3 \quad G'|G''| \Gamma_1, \Gamma_1' \vdash \Delta_1, \Delta_1'|\Gamma_2, \Gamma_2' \vdash \Delta_2, \Delta_2'|\Gamma_3, \Gamma_3' \vdash \Delta_3, \Delta_3'\]


\[
G|\Sigma, \Gamma_1 \vdash \Delta_1, \Pi \quad G'|\Sigma, \Gamma_2 \vdash \Delta_2, \Pi \quad G'|G''| \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Sigma \vdash \Pi
\]

(*, D.M. Gabbay, N. Olivetti. ”Cut-free proof systems for logics of weak excluded middle”. Soft Computing, 1998.)
HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic
aMALL + ((A → B) → B) → ((B → A) → A)
HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic

\[ \text{aMALL} + ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \]

Infinite-valued Łukasiewicz Logic

\[ \text{aMALL} + ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \]


Note: the interpretation of components (i.e. sequents) is changed!
HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic
aMALL + (\((A \rightarrow B) \rightarrow B\)) \rightarrow (\((B \rightarrow A) \rightarrow A\))


Note: the interpretation of components (i.e. sequents) is changed!

\(\Gamma \vdash \Delta\) is not interpreted as
\((AND \ A \in)\Gamma \rightarrow (OR \ B \in)\Delta\) but using the characteristic model \([-1, 0]\) of Łukasiewicz Logic.
HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic

\[ aMALL + ((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A) \]


Note: the interpretation of components (i.e. sequents) is changed!

\[ \Gamma \vdash \Delta \] is not interpreted as \( (AND A \in) \Gamma \rightarrow (OR B \in) \Delta \) but using the characteristic model \([-1, 0]\) of Łukasiewicz Logic.

I.e. if for all valuations \( v \) for \([-1, 0]\),

\[ \Sigma_{A \in \Gamma} v(A) \leq \Sigma_{B \in \Delta} v(B) \]

(where \( \Sigma_{A \in \emptyset} v(A) = 0 \))
HC for Łukasiewicz Logic

\[(ID)\] \( A \vdash A \) \hspace{1cm} \[(\Lambda)\] \( \vdash \) \hspace{1cm} \[(\bot)\] \( \bot \vdash A \)

Logical rules

\[
\frac{G|\Gamma, B \vdash A, \Delta|\Gamma \vdash \Delta}{G|\Gamma, A \rightarrow B \vdash \Delta}
\]

\[
\frac{G|\Gamma \vdash \Delta}{G|\Gamma, A \vdash B, \Delta}
\]

Internal structural rules

\[
\frac{G|\Gamma \vdash \Delta}{G|\Gamma, A \vdash \Delta} \quad (WL)
\]

\[
\frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \quad (M)
\]

External structural rules

\[
(\text{EW}), \quad (\text{EE}), \quad (\text{EC}) \quad \text{and} \quad (\text{S})
\]

Tableaux’03 – p.62/63
Concluding Remarks

- HC is a rather flexible framework

HC is a rather flexible framework, one can easily go beyond the propositional level by adding the usual quantifier rules (both for first-order and propositional quantifiers). One can (automatically) define analytic calculi for families of logics satisfying suitable properties. HC are a nice tool for analyzing and reasoning about proofs in the logics concerned.
Concluding Remarks

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HC are a nice tool for analyzing and reasoning about proofs in the logics concerned.