Hypersequent calculi for non classical logics

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A central task of logic in computer science is to provide *calculi* for large families of *non-classical logics*.

Uniform and Analytic Calculi

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- Analytic calculi are a basic prerequisite for developing automated reasoning methods. Moreover they may help to resolve such meta-logical issues as decidability, complexity, interpolation and admissibility of rules.

Sequent Calculi

G. Gentzen “Untersuchungen über das logische Schliessen I, II”. Mathematische Zeitschrift 1934

Sequent Calculi

Sequents

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

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Axioms

E.g., $A \vdash A$

Rules

- Logical
- Structural
- $\dfrac{\Gamma, A \vdash \Delta \quad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad (cut)$

Ex: LK calculus for CL

(sequents = multisets of formulas)

$$A \vdash A \qquad \text{(cut)}\ \frac{\Gamma_1 \vdash A, \Delta \quad A, \Gamma_2 \vdash}{\Gamma_1, \Gamma_2 \vdash B, \Delta}$$

$$(w, l)\ \frac{\Gamma \vdash C, \Delta}{\Gamma, A \vdash C, \Delta} \qquad (w, r)\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash C, \Delta}$$

$$(c, l)\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad (c, r)\ \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}$$

$$(\rightarrow, r)\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta} \qquad (\rightarrow, l)\ \frac{\Gamma \vdash A, \Delta \quad B, \Gamma \vdash}{\Gamma, A \rightarrow B \vdash \Delta}$$

Summary

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1. Introduction to Hypersequents
2. Adding quantifiers
3. Advanced Topics

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   Examples of HC: Intuitionistic and Classical Logic, LQ, Logics of bounded cardinality Kripke Models

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   Examples of HC: Intuitionistic and Classical Logic, LQ, Logics of bounded cardinality Kripke Models

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   Example: Gödel logics (propositional, first order and propositionally quantified)

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1. Introduction to Hypersequents

   Examples of HC: Intuitionistic and Classical Logic, LQ, Logics of bounded cardinality Kripke Models

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   Example: Gödel logics (propositional, first order and propositionally quantified)

3. Advanced Topics

   - cut-elimination
   - automated generation of hypersequent calculi (Examples: basic fuzzy logics and GIL)
   - variants of the hypersequent framework (Examples: Łukasiewicz logic and Logics of bounded depth Kripke Models)

Hypersequent Calculi

Hypersequent Calculi

A. Avron "A constructive analysis of RM". J. of Symbolic Logic. 1987.
(see also G. Pottinger "Uniform, cut-free formulation of T,S_4 and S_5, (abstract)". J. of Symbolic Logic. 1983.)

Hypersequent Calculi

are a simple and natural generalization of Sequent Calculi. A hypersequent is

$$\Gamma_1 \vdash \Pi_1 | \dots | \Gamma_n \vdash \Pi_n$$

where, for all $i = 1, \dots n$, $\Gamma_i \vdash \Pi_i$ is an ordinary sequent.

Hypersequent Calculi

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The symbol “$|$” is intended to denote disjunction at the meta-level.

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$\Gamma_i \vdash \Pi_i$ is called a component of the hypersequent.
The symbol "$|$" is intended to denote disjunction at the meta-level.
The above hypersequent is interpreted as

$$G_1 \vee .... \vee G_n$$

where G_i is the interpretation of the sequent $\Gamma_i \vdash \Pi_i$

Hypersequent Calculi

- general framework

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- independent of any particular semantics
- the rules for connectives are *standard*: the difference between the hypersequent calculi for various logics is in the set of their structural rules (Facilitate a better understanding and construction of the logics constructed within the framework.)

Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules

Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules

- Logical
- Structural
- Cut

Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules

Logical & Cut

Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules

Logical & Cut

E.g,

$$\frac{G \mid \Gamma \vdash A, \Delta \qquad G' \mid B, \Gamma \vdash \Delta}{G \mid G' \mid \Gamma, A \rightarrow B \vdash \Delta}\ (\rightarrow, l)$$

Hypersequent Calculi

Axioms E.g., $A \vdash A$

Rules

Logical & Cut

Structural

- Internal
- External

$$\frac{G}{G \mid \Gamma \vdash \Delta}\,(EW) \qquad \frac{G \mid \Gamma' \vdash \Delta' \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta \mid \Gamma' \vdash \Delta'}\,(EE)$$

$$\frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta}\,(EC)$$

(Hyper)sequent Calculus for IL

$$A \vdash A \qquad\qquad (\mathrm{cut})\ \frac{\Gamma_1 \vdash A \quad A, \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash B}$$

$$(w)\ \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad\qquad (c)\ \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

$$(\rightarrow, r)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad (\rightarrow, l)\ \frac{\Gamma \vdash A \quad B, \Gamma \vdash C}{\Gamma, A \rightarrow B \vdash C}$$

(Hyper)sequent Calculus for IL

$A \vdash A$

$$(\text{cut})\ \frac{G|\Gamma_1 \vdash A \quad G'|A, \Gamma_2 \vdash B}{G|G'|\Gamma_1, \Gamma_2 \vdash B}$$

$$(w)\ \frac{G|\Gamma \vdash C}{G|\Gamma, A \vdash C}$$

$$(c)\ \frac{G|\Gamma, A, A \vdash C}{G|\Gamma, A \vdash C}$$

$(EE), (EW)$

(EC)

$$(\rightarrow, r)\ \frac{G|\Gamma, A \vdash B}{G|\Gamma \vdash A \rightarrow B}$$

$$(\rightarrow, l)\ \frac{G|\Gamma \vdash A \quad G'|B, \Gamma \vdash C}{G|G'|\Gamma, A \rightarrow B \vdash C}$$

(Hyper)sequent Calculus for IL

A hypersequent

$$\Gamma_1 \vdash A_1 \mid \ldots \mid \Gamma_n \vdash A_n$$

is provable in the hypersequent calculus for IL if and only if there exists $i \in \{1, \ldots, n\}$ such that LJ proves

$$\Gamma_i \vdash A_i$$

Some Structural Rules

- $$\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash |\Gamma' \vdash A}\ (cl)$$

- $$\frac{G \mid \Gamma, \Gamma' \vdash}{G|\Gamma \vdash |\Gamma' \vdash}\ (lq)$$

- $$\frac{G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma_1' \vdash A'}{G|G'|\Gamma, \Gamma_1' \vdash A|\Gamma', \Gamma_1 \vdash A'}\ (com)$$

Hypersequents and Parallelism

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- Avron suggested that a hypersequent can be thought of as a multiprocess.
  (A. Avron “Hypersequents, Logical Consequence and Intermediate Logics for Concurrency”. Annals of Mathematics and Artificial Intelligence. 1991)

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$$\frac{G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma'_1 \vdash A'}{G|G'|\Gamma, \Gamma'_1 \vdash A|\Gamma', \Gamma_1 \vdash A'} \; (com)$$

Hypersequents and Parallelism

- Avron suggested that a hypersequent can be thought of as a multiprocess. (A. Avron “Hypersequents, Logical Consequence and Intermediate Logics for Concurrency”. Annals of Mathematics and Artificial Intelligence. 1991)
- Fermüller characterized hypersequent calculi by (suitable) games of communicating parallel dialogues (C.G. Fermüller “Parallel Dialogue Games and Hypersequents for Intermediate Logics”. Tableaux 2003)

Example: CL

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$A \vdash A$

$$(\mathrm{cut}) \quad \frac{G|\Gamma_1 \vdash A, \Delta \quad G'|A, \Gamma_2 \vdash B}{G|G'|\Gamma_1, \Gamma_2 \vdash B, \Delta}$$

$$(w, l) \quad \frac{G|\Gamma \vdash C, \Delta}{G|\Gamma, A \vdash C, \Delta}$$

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$$(\rightarrow, r) \quad \frac{G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \rightarrow B, \Delta}$$

$$(\rightarrow, l) \quad \frac{G|\Gamma \vdash A, \Delta \quad G'|B, \Gamma \vdash}{G|G'|\Gamma, A \rightarrow B \vdash \Delta}$$

Example: CL

Hypersequent calculus for IL +

$$\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash |\Gamma' \vdash A}\ (cl)$$

= (cut-free) Hypersequent calculus for Classical Logic

*, D.M. Gabbay, N. Olivetti. ”Cut-free proof systems for logics of weak excluded middle”. Soft Computing, 1998.

Example: CL

Hypersequent calculus for IL +

$$\frac{G|\Gamma, \Gamma' \vdash A}{G|\Gamma \vdash |\Gamma' \vdash A}\ (cl)$$

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Idea behind the rule: let $\Gamma' = \emptyset$ and $\Gamma = A$.

$$\frac{G|A \vdash A}{G|A \vdash | \vdash A}\ (cl)$$

The (cl) rule performs a case-analysis, namely it says that for every formula A and evaluation v, $v(A) = 0$ or $v(A) = 1$

Example: CL

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Soundness & Completeness

Hilbert system: set of axiom schemes together with one rule of inference: modus ponens

$$\frac{A \quad A \rightarrow B}{B}$$

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Soundness

We prove that the interpretation of axioms (i.e. $A \to A$) is provable in the Hilbert system for CL and that for each rule that whenever the Hilbert system for CL derives the interpretations of its premises, it derives the interpretation of its conclusion too. E.g., For (cl): if $\underline{G} \vee ((\bigwedge \Gamma \wedge \bigwedge \Gamma') \to A)$ is provable in the H. system for CL, so is $\underline{G} \vee \neg \bigwedge \Gamma \vee (\bigwedge \Gamma') \to A$.

Example: CL

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Completeness

Modus Ponens corresponds to the derivability of $A, A \rightarrow B \vdash B$ and the cut rule. It thus suffices to show that all the axioms of the Hilbert system for CL are derivable in the calculus.

Example: CL

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= (cut-free) Hypersequent calculus for Classical Logic

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash\ |\ \vdash A}\ \text{(cl)}}{\vdash \neg A \mid\ \vdash A}\ (\neg,\text{r})}{\neg\neg A \vdash\ |\ \vdash A}\ (\neg,\text{l})}{\neg\neg A \vdash A \mid \neg\neg A \vdash A}\ \text{2x(w)}}{\neg\neg A \vdash A}\ \text{(EC)}}{\vdash \neg\neg A \to A}\ (\to,\text{r})$$

Example: LQ

LQ is the intermediate logic semantically characterized by the class of all finite and rooted posets with a single final element.

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LQ is the intermediate logic semantically characterized by the class of all finite and rooted posets with a single final element.
A Hilbert-style axiomatization for LQ is obtained by adding to that of IL ,i.e., e.g.

$$\begin{array}{ll} \bot \to A & A \to (B \to A) \\ (A \wedge B) \to A & (A \wedge B) \to B \\ A \to (A \vee B) & B \to (A \vee B) \end{array}$$

$$(A \to (B \to C)) \to (B \to (A \to C))$$
$$(A \to B) \to ((C \to A) \to (C \to B))$$
$$A \to (B \to (A \wedge B))$$
$$(A \to B) \to ((C \to B) \to ((A \vee C) \to B))$$
$$(A \to (A \to B)) \to (A \to B)$$

Example: LQ

LQ is the intermediate logic semantically characterized by the class of all finite and rooted posets with a single final element.
A Hilbert-style axiomatization for LQ is obtained by adding to that of IL axiom

$$\neg A \vee \neg\neg A$$

(V.A. Jankov "The calculus of the weak “law of excluded middle”". Mathematics of the USSR. 1968)

A Hypersequent calculus for LQ

Sequent Formulation:
T. Hosoi. ”Gentzen-type Formulation of the Propositional Logic LQ”. Studia Logica. 1988.

A Hypersequent calculus for LQ

Hypersequent calculus for IL +

$$\frac{G \mid \Gamma, \Gamma' \vdash}{G|\Gamma \vdash |\Gamma' \vdash} \ (lq)$$

= (cut-free) Hypersequent calculus for LQ
(*, D.M. Gabbay, N. Olivetti. ”Cut-free Proof Systems for Logics of Weak Excluded Middle”. Soft Computing. 1998)

A Hypersequent calculus for LQ

Hypersequent calculus for IL +

$$\frac{G \mid \Gamma, \Gamma' \vdash}{G|\Gamma \vdash |\Gamma' \vdash}\ (lq)$$

= (cut-free) Hypersequent calculus for LQ

$$\begin{array}{cl}
A \vdash A & \\ \hline
A, \neg A \vdash & (\neg,\mathrm{l}) \\ \hline
A \vdash |\neg A \vdash & (\mathrm{lq}) \\ \hline
A \vdash | \vdash \neg\neg A & (\neg,\mathrm{r}) \\ \hline
\vdash \neg A| \vdash \neg\neg A & (\neg,\mathrm{r}) \\ \hline
\vdash \neg A| \vdash \neg A \vee \neg\neg A & (\vee,\mathrm{r}) \\ \hline
\vdash \neg A \vee \neg\neg A| \vdash \neg A \vee \neg\neg A & (\vee,\mathrm{r}) \\ \hline
\vdash \neg A \vee \neg\neg A & (\mathrm{EC})
\end{array}$$

Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most k nodes.

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Family of logics semantically characterized by the class of trees containing at most k nodes.
A Hilbert style axiomatization is given by

$$IL + \{p_0 \vee (p_0 \longrightarrow p_1) \vee \ldots \vee (p_0 \wedge \ldots \wedge p_{k-1} \longrightarrow p_k)\}$$

Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most k nodes. Particular cases: $k = 1$ Classical Logic, $k = 2$ SM logic.

Ex: Logics of bounded Kripke models

Family of logics semantically characterized by the class of trees containing at most k nodes.

Hypersequent Calculus
(for $k \geq 1$) Hypersequent calculus for IL +

$$\frac{\ldots \quad G_{i,j} \mid \Gamma_i, \Gamma_j \vdash A_i \quad \ldots}{G_{0,1} \mid \ldots \mid G_{k-1,k} \mid \Gamma_0 \vdash A_0 \mid \ldots \mid \Gamma_{k-1} \vdash A_{k-1} \mid \Gamma_k}$$

for every i, j such that $0 \leq i \leq k-1$ and $i+1 \leq j \leq k$.

(*, M. Ferrari. ”Hypersequent calculi for some intermediate logics with bounded Kripke models". J. of Logic and Computation. 2001)

Summary

1. Introduction to Hypersequents
2. Adding quantifiers

   Example: Gödel logics (propositional, first order and propositionally quantified)
3. Advanced Topics

Adding Quantifiers

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Two different forms of quantification:

- first-order quantifiers
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- propositional quantifiers
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M. Baaz, *, C. Fermüller. ”Hypersequent Calculi for Gödel Logics — a Survey”. J. of Logic and Computation. 2003

Gödel logic(s)

Gödel ’33

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$$sm(P_1 \bigcup P) = sm(P_2 \bigcup P) \quad \forall P$$

Gödel logic(s)

Gödel ’33

- related to relevance logics (Dunn and Meyer ’71)
- employed to investigate the provability logic of Heyting arithmetic (Visser ’82)
- used to model strong equivalence between logic programs (Lifschitz et al.2002)
- one of the main formalizations of fuzzy logic (Hajek’98)

Gödel logic G_∞

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Axiomatization

$$\mathrm{G}_\infty = IL + (A \to B) \vee (B \to A)$$

(M. Dummett ”A Propositional Logic with Denumerable Matrix” J. of Symbolic Logic. 1959)

Gödel logic G_∞

G_∞ is characterized by the class of all rooted linearly ordered Kripke models.

Axiomatization

$$\mathrm{G}_\infty = IL + (A \rightarrow B) \vee (B \rightarrow A)$$

Many-valued Semantics

v : Propositions $\rightarrow \ [0,1]$

$$v(A \wedge B) = \min\{v(A), v(B)\}$$

$$v(A \vee B) = \max\{v(A), v(B)\}$$

$$v(A \rightarrow B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ \text{v(B)} & \text{otherwise} \end{cases} \qquad v(\bot) = 0$$

Hypersequent Calculus for G_∞

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Cut free Sequent calculus

O. Sonobo. ”A Gentzen-type formulation for some intermediate propositional logics”. J. of Tsuda College. 1975

Hypersequent Calculus for G_∞

Hypersequent Calculus for Intuitionistic Logic +

$$\frac{G \mid \Gamma, \Gamma' \vdash A \quad G' \mid \Gamma_1, \Gamma_1' \vdash A'}{G \mid G' \mid \Gamma, \Gamma_1 \vdash A \mid \Gamma', \Gamma_1' \vdash A'} \ (com)$$

A. Avron "Hypersequents, Logical Consequence and Intermediate Logics for Concurrency". Annals of Mathematics and Artificial Intelligence. 1991

Hypersequent Calculus for G_∞

Hypersequent Calculus for Intuitionistic Logic +

$$\frac{G \mid \Gamma, \Gamma' \vdash A \qquad G' \mid \Gamma_1, \Gamma_1' \vdash A'}{G \mid G' \mid \Gamma, \Gamma_1 \vdash A \mid \Gamma', \Gamma_1' \vdash A'}\ (com)$$

Example (completeness)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \vdash B \mid B \vdash A}\,(\text{com})}{A \vdash B \mid\ \vdash B \to A}\,(\to,\text{r})}{\vdash A \to B \mid\ \vdash B \to A}\,(\to,\text{r})}{\vdash A \to B \mid\ \vdash (A \to B) \vee (B \to A)}\,(\vee_\text{i},\text{r})}{\vdash (A \to B) \vee (B \to A) \mid\ \vdash (A \to B) \vee (B \to A)}\,(\vee_\text{i},\text{r})}{\vdash (A \to B) \vee (B \to A)}\,(\text{EC})$$

Gödel logic G_{k+1}

Infinite valued Gödel logic is characterized by the class of all rooted linearly ordered Kripke models

Axiomatization

$$\mathrm{G}_\infty = IL + (A \to B) \vee (B \to A)$$

Many-valued Semantics

v : Propositions $\to$ $[0,1]$

$$v(A \wedge B) = \min\{v(A), v(B)\}$$

$$v(A \vee B) = \max\{v(A), v(B)\}$$

$$v(A \to B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ \text{v(B)} & \text{otherwise} \end{cases} \qquad v(\perp) = 0$$

Gödel logic G_{k+1}

G_{k+1} is characterized by the class of all rooted linearly ordered Kripke models *with at most k worlds*

Axiomatization

$$\mathrm{G}_\infty = IL + (A \to B) \vee (B \to A)$$

$$+ A_1 \vee (A_1 \to A_2) \vee \ldots \vee (A_1 \wedge \ldots \wedge A_k \to A_{k+1})$$

Many-valued Semantics

v : Propositions $\longrightarrow$ $[0,1]$ $\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$

$$v(A \wedge B) = \min\{v(A), v(B)\}$$

$$v(A \vee B) = \max\{v(A), v(B)\}$$

$$v(A \to B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ \text{v(B)} & \text{otherwise} \end{cases} \qquad v(\bot) = 0$$

Hypersequent Calculi for G_{k+1}

(*, M. Ferrari Hypersequent calculi for some intermediate logics with bounded Kripke models". J. of Logic and Computation. 2001)

Hypersequent Calculus for Intuitionistic Logic +

Hypersequent Calculi for G_{k+1}

Hypersequent Calculus for Intuitionistic Logic +

$$\frac{G \mid \Gamma, \Gamma' \vdash A \qquad G' \mid \Gamma_1, \Gamma_1' \vdash A'}{G \mid G' \mid \Gamma, \Gamma_1 \vdash A \mid \Gamma', \Gamma_1' \vdash A'}\ (com)$$

$$+$$

$$\frac{\ldots \qquad G_{i,j} \mid \Gamma_i, \Gamma_j \vdash A_i \qquad \ldots}{G_{0,1} \mid \ \ldots \ \mid G_{k-1,k} \mid \Gamma_0 \vdash A_0 \mid \ \ldots \ \mid \Gamma_{k-1} \vdash A_{k-1} \mid \Gamma_k}$$

for every i, j such that $0 \leq i \leq k-1$ and $i+1 \leq j \leq k$.

Hypersequent Calculi for G_{k+1}

Hypersequent Calculus for Intuitionistic Logic +

$$\frac{G_1 \mid \Gamma_1, \Gamma_2 \vdash A_1 \quad \ldots \quad G_k \mid \Gamma_k, \Gamma_{k+1} \vdash A_k}{G_1 \mid \ldots \mid G_k \mid \Gamma_1 \vdash A_1 \mid \ldots \mid \Gamma_k \vdash A_k \mid \Gamma_{k+1} \vdash}$$

Example: HC for G_3

Hypersequent Calculus for Intuitionistic Logic +

$$\frac{G_1 \mid \Gamma_1, \Gamma_2 \vdash A_1 \quad G_2 \mid \Gamma_2, \Gamma_3 \vdash A_2}{G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid \Gamma_3 \vdash} \ (G_3)$$

Example: HC for G_3

Hypersequent Calculus for Intuitionistic Logic +

$$\frac{G_1 \mid \Gamma_1, \Gamma_2 \vdash A_1 \qquad G_2 \mid \Gamma_2, \Gamma_3 \vdash A_2}{G_1 \mid G_2 \mid \Gamma_1 \vdash A_1 \mid \Gamma_2 \vdash A_2 \mid \Gamma_3 \vdash}\ (G_3)$$

Ex: $(P := A_1 \vee (A_1 \to A_2) \vee (A_1 \wedge A_2 \to A_3))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A_1 \vdash A_1 \qquad \dfrac{\dfrac{A_2 \vdash A_2}{A_1 \wedge A_2 \vdash A_2}\ (\wedge,\text{l})}{A_1, A_1 \wedge A_2 \vdash A_2}\ (\text{w})}{\vdash A_1 \mid A_1 \vdash A_2 \mid A_1 \wedge A_2 \vdash}\ (\text{G}_3)}{\vdash A_1 \mid A_1 \vdash A_2 \mid A_1 \wedge A_2 \vdash A_3}\ (\text{w})}{\vdash A_1 \mid \vdash A_1 \to A_2 \mid \vdash A_1 \wedge A_2 \to A_3}\ \text{2x}(\to,\text{r})}{\vdash P \mid \vdash P \mid \vdash P}\ \text{3x}(\vee,\text{r})}{\vdash P}\ \text{2x(EC)}$$

First Order Gödel logic

Intuitionistic Fuzzy Logic (Takeuti and Titani. JSL. 1984)

First Order Gödel logic

is characterized by the class of all rooted linearly ordered Kripke models with *constant domains*.

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Many valued semantics

An interpretation I= (domain D, valuation function v_I) (P^n mapped into $D^n \to [0,1]$).

$$v_I((\forall x)A(x)) = \inf\{v_{I'}(A(x))\}$$
$$v_I((\exists x)A(x)) = \sup\{v_{I'}A(x))\}$$

(where v_I' is exactly as v_I with the possible exception of the domain element assigned to x).

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$$\mathrm{G}_\infty = IL + (A \rightarrow B) \vee (B \rightarrow A)$$

+ Ax for quant. + $\forall x(A(x) \vee B) \rightarrow (\forall x A(x)) \vee B$

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? + ? (Takeuti and Titani)

$$\frac{\Gamma \vdash C \vee (A \to p) \vee (p \to B)}{\Gamma \vdash C \vee (A \to B)}$$

where p is does not occur in the conclusion.

Hypersequent Calculus for FO Gödel Logic

M. Baaz and R. Zach "Hypersequents and the proof theory of intuitionistic fuzzy logic". Proc. of CSL’2000

Hypersequent Calculus for propositional Gödel Logic

+

Hypersequent Calculus for FO Gödel Logic

Hypersequent Calculus for propositional Gödel Logic

+

$$\frac{G \mid A(t), \Gamma \vdash B}{G \mid (\forall x)A(x), \Gamma \vdash B}\ (\forall, l) \qquad \frac{G \mid \Gamma \vdash A(a)}{G \mid \Gamma \vdash (\forall x)A(x)}\ (\forall, r)$$

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Hypersequent Calculus for FO Gödel Logic

Hypersequent Calculus for propositional Gödel Logic

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where in $(\forall, r)$ and $(\exists, l)$ the free variable a must not occur in the lower *hypersequent*.

HC for FO Gödel Logic

Ex.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(a)\vdash A(a)\qquad \dfrac{A(a)\vdash A(a)\qquad B\vdash B}{B\vdash A(a)\mid A(a)\vdash B}\,(com)\qquad B\vdash B}{A(a)\vee B\vdash A(a)\mid A(a)\vee B\vdash B}\,2x(\vee}{(\forall x)(A(x)\vee B)\vdash A(a)\mid(\forall x)(A(x)\vee B)\vdash B}\,2x(\forall,l)}{(\forall x)(A(x)\vee B)\vdash(\forall x)A(x)\mid(\forall x)(A(x)\vee B)\vdash B}\,(\forall,r)}{(\forall x)(A(x)\vee B)\vdash(\forall x)A(x)\vee B\mid(\forall x)(A(x)\vee B)\vdash(\forall x)A(x)}}{(\forall x)(A(x)\vee B)\vdash(\forall x)A(x)\vee B}}{\vdash(\forall x)(A(x)\vee B)\rightarrow((\forall x)A(x)\vee B)}\,(\rightarrow,r)$$

Midsequent Theorem

In Gentzen's **LK** — as a consequence of cut-elimination — a separation between propositional and quantificational inferences can be achieved in deriving a prenex sequent (midsequent theorem). See, e.g. G. Takeuti. Proof Theory. 1987

Midsequent Theorem

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(*Sketch of Proof*)

Proceeds by induction on the *order* of a derivation.

From

$$\frac{\text{quantifier rule}}{\text{logical rule}}$$

To

$$\frac{\text{logical rule}}{\text{quantifier rule}}$$

Midsequent Theorem

In Gentzen's **LK** — as a consequence of cut-elimination — a separation between propositional and quantificational inferences can be achieved in deriving a prenex sequent (midsequent theorem). This result does not hold for **LJ**.
E.g.

$$\dfrac{\dfrac{\Gamma, Y \vdash A(a)}{\Gamma, Y \vdash (\forall x)A(x)}\,(\forall,\mathrm{r}) \qquad \dfrac{\Gamma, X \vdash A(b)}{\Gamma, X \vdash (\forall x)A(x)}\,(\forall,\mathrm{r})}{\Gamma, X \vee Y \vdash (\forall x)A(x)}\,(\vee,\mathrm{l})$$

FO Gödel Logic and Mid(hyper)sequent Theorem

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Any derivation in the HC for FO Gödel Logic of a prenex hypersequent can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.

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Any derivation in the HC for FO Gödel Logic of a prenex hypersequent can be transformed into one in which no propositional rule is applied below any application of a quantifier rule.
(*Sketch of Proof*)
Proceeds by induction on the *order* of a derivation.
Note that the following rule is derivable in the HC for FO Gödel Logic.

$$\frac{G \mid A, \Gamma \vdash C_1 \quad G \mid B, \Gamma \vdash C_2}{G \mid A \vee B, \Gamma \vdash C_1 \mid A \vee B, \Gamma \vdash C_2}\ (\vee', l)$$

Mid(hyper)sequent Theorem – Sketch of Proof

We replace all the applications of $(\vee, l)$ by applications of $(\vee', l)$.

Mid(hyper)sequent Theorem – Sketch of Proof

We replace all the applications of $(\vee, l)$ by applications of $(\vee', l)$. Therefore

$$\dfrac{\dfrac{\Gamma, Y \vdash A(a)}{\Gamma, Y \vdash (\forall x)A(x)}(\forall\text{,r}) \quad \dfrac{\Gamma, X \vdash A(b)}{\Gamma, X \vdash (\forall x)A(x)}(\forall\text{,r})}{\Gamma, X \vee Y \vdash (\forall x)A(x)}(\vee\text{,l})$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\Gamma, Y \vdash A(a)}{\Gamma, Y \vdash (\forall x)A(x)}(\forall\text{,r}) \quad \dfrac{\Gamma, X \vdash A(b)}{\Gamma, X \vdash (\forall x)A(x)}(\forall\text{,r})}{\Gamma, X \vee Y \vdash (\forall x)A(x) \mid \Gamma, X \vee Y \vdash (\forall x)A(x)}(\vee'\text{,l})}{\Gamma, X \vee Y \vdash (\forall x)A(x)}(\text{EC})$$

Mid(hyper)sequent Theorem – Sketch of Proof

We replace all the applications of $(\vee, l)$ by applications of $(\vee', l)$.

$$\dfrac{\dfrac{\dfrac{\Gamma, Y \vdash A(a)}{\Gamma, Y \vdash (\forall x)A(x)}\,(\forall,\mathrm{r}) \qquad \dfrac{\Gamma, X \vdash A(b)}{\Gamma, X \vdash (\forall x)A(x)}\,(\forall,\mathrm{r})}{\Gamma, X \vee Y \vdash (\forall x)A(x) \mid \Gamma, X \vee Y \vdash (\forall x)A(x)}\,(\vee',\mathrm{l})}{\Gamma, X \vee Y \vdash (\forall x)A(x)}\,(\mathrm{EC})$$

Can be transformed into

$$\dfrac{\dfrac{\dfrac{\Gamma, Y \vdash A(a) \qquad \Gamma, X \vdash A(b)}{\Gamma, X \vee Y \vdash A(a) \mid \Gamma, X \vee Y \vdash A(b)}\,(\vee',\mathrm{l})}{\Gamma, X \vee Y \vdash (\forall x)A(x) \mid \Gamma, X \vee Y \vdash (\forall x)A(x)}\,2\mathrm{x}(\forall,\mathrm{r})}{\Gamma, X \vee Y \vdash (\forall x)A(x)}\,(\mathrm{EC})$$

FO Gödel Logic and the (TT) rule

$$\frac{\Gamma \vdash C \vee (A \to p) \vee (p \to B)}{\Gamma \vdash C \vee (A \to B)}$$

where p is does not occur in the conclusion.
This rule, expressing the density of the ordered set of truth-values, was used by Takeuti and Titani to axiomatize first-order Gödel logic.

FO Gödel Logic and the (TT) rule

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This rule, expressing the density of the ordered set of truth-values, was used by Takeuti and Titani to axiomatize first-order Gödel logic.

Takano (1984) posed the question whether a syntactical elimination of this rule is also possible. The hypersequent calculus for FO Gödel Logic allows one to give a positive answer to this question.

Quantified Propositional Gödel logic

Generalization of propositional Gödel logic obtained by adding quantifiers over propositional variables

Quantified Propositional Gödel logic

$$v((\exists p)A) = \sup\{v[w/p](A) : w \in [0,1]\}$$
$$v((\forall p)A) = \inf\{v[w/p](A) : w \in [0,1]\}$$

Quantified Propositional Gödel logic

$$G_\infty = IL + (A \to B) \vee (B \to A) +$$

$$\frac{Z[a] \to Y}{((\exists q)Z[q]) \to Y}\ (R\exists) \qquad \frac{Y \to Z[a]}{Y \to (\forall q)Z[q]}\ (R\forall)$$

where a does not occur in Y.

$$A[X] \to (\exists q)A[q] \qquad ((\forall q)A[q]) \to A[X]$$

$$\vee{-}Shift: \quad ((\forall q)(A \vee B)) \to (A \vee (\forall q)B)$$

$$Density: \quad [(\forall q')((A \to q') \vee (q' \to B))] \to (A \to B)$$

where q does not occur in A and q' occurs neither in A nor in B.

Hypersequent Calculus for QP Gödel Logic

M. Baaz and C. Fermüller and H. Veith "An Analitic Calculus for Quantified Propositional Gödel Logic". Proceedings of Tableaux 2000

Hypersequent Calculus for propositional Gödel Logic

+

Hypersequent Calculus for QP Gödel Logic

Hypersequent Calculus for propositional Gödel Logic

+

$$\frac{G \mid A[X], \Gamma \vdash B}{G \mid (\forall q)A[q], \Gamma \vdash B}\ (\forall, l)^0 \qquad \frac{G \mid \Gamma \vdash A[a]}{G \mid \Gamma \vdash (\forall q)A[q]}\ (\forall, r)^0$$

$$\frac{G \mid A[a], \Gamma \vdash B}{G \mid (\exists q)A[q], \Gamma \vdash B}\ (\exists, l)^0 \qquad \frac{G \mid \Gamma \vdash A[X]}{G \mid \Gamma \vdash (\exists q)A[q]}\ (\exists, r)^0$$

+

$$\frac{G \mid \Pi \vdash p \mid p, \Gamma \vdash C}{G \mid \Pi, \Gamma \vdash C}\ (tt)$$

Summary

1. Introduction to Hypersequents
2. Adding quantifiers
3. Advanced Topics
    - cut-elimination
    - automated generation of hypersequent calculi (Examples: basic fuzzy logics and global intuitionistic logic)
    - variants of the hypersequent framework (Examples: Łukasiewicz logic and Logics of bounded depth Kripke Models)

Cut-elimination in HC

Cut-elimination in HC

Gentzen Method

Proceeds by eliminating the uppermost cut by a double induction on the complexity of the cut formula and on the sum of its left and right ranks; where the right (left) rank of a cut is the number of consecutive (hyper)sequents containing the cut formula, counting upward from the right (left) upper sequent of the cut.

Cut-elimination in HC

If $G' \mid \Gamma \vdash A$ and $H' \mid \Gamma', A^{(n)} \vdash B$ are cut-free provable in a hypersequent calculus, so is $G' \mid H' \mid \Gamma, \Gamma' \vdash B$.

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Pb. with Gentzen's method

$$\dfrac{\dfrac{\Gamma, A \vdash B | \Gamma, A \vdash B}{\Gamma, A \vdash B}\,(\mathrm{EC}) \qquad \Sigma \vdash A}{\Gamma, \Sigma \vdash B}\,(\mathrm{cut})$$

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If $G' \mid \Gamma \vdash A$ and $H' \mid \Gamma', A^{(n)} \vdash B$ are cut-free provable in a hypersequent calculus, so is $G' \mid H' \mid \Gamma, \Gamma' \vdash B$.

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$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B | \Gamma, A \vdash B \qquad \Sigma \vdash A}{\Gamma, \Sigma \vdash B \mid \Gamma, A \vdash B}\,(\mathrm{cut}) \qquad \Sigma \vdash A}{\Gamma, \Sigma \vdash B \mid \Gamma, \Sigma \vdash B}\,(\mathrm{cut})}{\Gamma, \Sigma \vdash B}\,(\mathrm{EC})$$

Cut-elimination

Solution n. 1:
Use generalized cut rules

Cut-elimination

E.g. If $G \mid \Gamma_1 \vdash A \mid \ldots \mid \Gamma_n \vdash A$ and $H \mid \Sigma_1, A^{n_1} \vdash B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is

$H \mid G \mid \Gamma_1 \vdash \ \mid \ldots \mid \Gamma_n \vdash \ \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k$.

Cut-elimination

E.g. If $G \mid \Gamma_1 \vdash A \mid \ldots \mid \Gamma_n \vdash A$ and $H \mid \Sigma_1, A^{n_1} \vdash B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is
$H \mid G \mid \Gamma_1 \vdash \ \mid \ldots \mid \Gamma_n \vdash \ \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k$.
It holds in Classical Logic

Cut-elimination

If $\underline{G} := G \mid \Gamma_1 \vdash A \mid \ldots \mid \Gamma_n \vdash A$ and $\underline{H} := H \mid \Sigma_1, A^{n_1} \vdash B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $H \mid \Gamma_1 \vdash\ \mid \ldots \mid \Gamma_n \vdash\ \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k$.

$$\dfrac{\dfrac{\underline{H} \mid \Sigma_k, A^{n_k} \vdash B_k}{\underline{H}}\,(\mathrm{EC}) \qquad\qquad \underline{G}}{H \mid G \mid \Gamma_1 \vdash\ \mid \ldots \mid \Gamma_n \vdash\ \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k}\,(\mathrm{cut})$$

$$\dfrac{\dfrac{\underline{H} \mid \Sigma_k, A^{n_k} \vdash B_k \qquad\qquad \underline{G}}{H \mid G \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k \mid \Sigma_k \vdash B_k \mid \Gamma_1 \vdash\ \mid \ldots \mid \Gamma_n \vdash}\,(\mathrm{cut})}{H \mid G \mid \Gamma_1 \vdash\ \mid \ldots \mid \Gamma_n \vdash\ \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k}\,(\mathrm{EC})$$

Cut-elimination

For Classical Logic

If $G \mid \Gamma_1 \vdash A \mid \ldots \mid \Gamma_n \vdash A$ and $H \mid \Sigma_1, A^{n_1} \vdash B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $G \mid H \mid \Gamma_1 \vdash \ \mid \ldots \mid \Gamma_n \vdash \ \mid \Sigma_1 \vdash B_1 \mid \ldots \mid \Sigma_k \vdash B_k$.

For Gödel Logic

If $G \mid \Gamma_1 \vdash A \mid \ldots \mid \Gamma_n \vdash A$ and $H \mid \Sigma_1, A^{n_1} \vdash B_1 \mid \ldots \mid \Sigma_k, A^{n_k} \vdash B_k$ are cut-free provable, so is $G \mid H \mid \Gamma, \Sigma_1 \vdash B_1 \mid \ldots \mid \Gamma, \Sigma_k \vdash B_k$, where $\Gamma = \Gamma_1, \ldots, \Gamma_n$.

Cut-elimination

W.W. Tait. Normal derivability in classical logic. In: The Sintax and Semantics of infinitary Languages. LNM. 1968.
K. Schütte. *Beweistheorie*. Springer Verlag. 1960.

Cut-elimination

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Schütte-Tait Method
Proceeds by eliminating the largest cut (w.r.t. the number of connectives and quantifiers).

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Schütte-Tait Method
Proceeds by eliminating the largest cut (w.r.t. the number of connectives and quantifiers).

A cut is not shifted upward but simply reduced (replaced by smaller cuts) in the place in which the cut-formula is introduced.

Cut-elimination

M. Baaz, *. A Schütte-Tait style cut-elimination proof for first-order Gödel logic. Proceedings of Tableaux 2002.

Cut-elimination

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid \Gamma' \vdash X\end{array} \qquad \begin{array}{c}\vdots\, d'\\ H \mid \Gamma, X \vdash B\end{array}}{G \mid H \mid \Gamma, \Gamma' \vdash B}\,(\mathrm{cut})$$

Cut-elimination

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid \Gamma' \vdash X\end{array} \qquad \begin{array}{c}\vdots\, d'\\ H \mid \Gamma, X \vdash B\end{array}}{G \mid H \mid \Gamma, \Gamma' \vdash B}\ (\mathrm{cut})$$

Two cases:

- atomic cut
- non atomic cut

Cut-elimination

$$\dfrac{\begin{array}{c}\vdots\; d\\ G \mid \Gamma' \vdash X^*\end{array} \qquad \begin{array}{c}\vdots\; d'\\ H \mid \Gamma, X^* \vdash B\end{array}}{G \mid H \mid \Gamma, \Gamma' \vdash B}\;(\mathrm{cut})$$

Two cases:

- atomic cut
- non atomic cut

Cut-elimination

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid \Gamma' \vdash X^*\end{array} \qquad \begin{array}{c}\vdots\, d'\\ H \mid \Gamma, X^* \vdash B\end{array}}{G \mid H \mid \Gamma, \Gamma' \vdash B}\,(\mathrm{cut})$$

Two cases:

- atomic cut
  Replace X^* in d' by Γ'.
  Two possibilities:
  - X^* originates in a weakening rule.
  - X^* originates in an axiom $X^* \vdash X$.

  We get a proof of $G \mid H \mid \Gamma, \Gamma' \vdash B$ starting from a proof of $G \mid \Gamma' \vdash X$
- non atomic cut

Cut-elimination

$$\dfrac{\begin{matrix}\vdots\; d\\ G \mid \Gamma' \vdash X^*\end{matrix}\qquad \begin{matrix}\vdots\; d'\\ H \mid \Gamma, X^* \vdash B\end{matrix}}{G \mid H \mid \Gamma, \Gamma' \vdash B}\,(\mathrm{cut})$$

Two cases:

- atomic cut
- non atomic cut
  1. one first inverts (reduces the complexity of the cut formula in) one of the two sides of the cut; (Inversion Lemma)
  2. the cut is then reduced (replaced by smaller cuts) in the place in which the cut formula is introduced; (Reduction Lemma).

Cut-elimination

Inversion Lemma

Cut-elimination

Inversion Lemma
(reduces the complexity of the cut formula on the "invertible" side)

Cut-elimination

Inversion Lemma

1. If d is a derivation of $G \mid \Gamma, A \vee B \vdash C$ one can find $d_1,\ G \mid \Gamma, A \vdash C$ and $d_2,\ G \mid \Gamma, B \vdash C$
2. If $d,\ G \mid \Gamma \vdash A \wedge B$ then one can find $d_1,\ G \mid \Gamma \vdash A$ and $d_2,\ G \mid \Gamma \vdash B$
3. If $d, G \mid \Gamma \vdash A \rightarrow B$ one can find $d_1,\ G \mid \Gamma, A \vdash B$
4. If $d, G \mid \Gamma, \exists x A(x) \vdash C$ one can find $d_1,\ G \mid \Gamma, A(a) \vdash C$
5. If $d, G \mid \Gamma \vdash \forall x A(x)$ one can find $d_1,\ G \mid \Gamma \vdash A(a)$

such that $c(d_i) \leq c(d)$ and the $l(d_i) \leq l(d)$, $i = 1, 2$.

Cut-elimination

Reduction Lemma

E.g.:

$$\dfrac{\begin{array}{c}\vdots\\ G \mid \Sigma \vdash B \to D\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ G' \mid \Psi \vdash B\end{array} \quad \begin{array}{c}\vdots\\ G' \mid \Psi, D \vdash C'\end{array}}{}\,(\to,\mathrm{l})\\ \vdots\\ H \mid \Gamma, B \to D \vdash C\end{array}}{G \mid H \mid \Gamma, \Sigma \vdash C}\,(\mathrm{cut})$$

Using $G \mid \Sigma, B \vdash D$ (Inversion Lemma)

Cut-elimination

Reduction Lemma

E.g.:

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Using $G \mid \Sigma, B \vdash D$ (Inversion Lemma)

Automated Generation of HC

Logic L
(cut free (hyper)sequent calculus)

Automated Generation of HC

Logic L
(cut free (hyper)sequent calculus)

$+$

”certain” properties $(\star)$

Automated Generation of HC

Logic L
(cut free (hyper)sequent calculus)

$+$

”certain” properties $(\star)$

$\Longrightarrow$

Logic $L^\star$

(cut free hypersequent sequent calculus)

Automated Generation of HC

Properties: (We consider logics as characterized by their Hilbert style systems)

1. $(A \to B) \vee (B \to A)$
2. modality Δ s.t.

$$(1)\,\Delta A \to A \qquad (2)\,\Delta A \to \Delta\Delta A$$
$$(3)\,\Delta A \vee \neg\Delta A \quad (4)\,\Delta(A \vee B) \to \Delta A \vee \Delta B$$

$$(5)\,\Delta(A \to B) \to (\Delta A \to \Delta B)$$

$$\frac{A}{\Delta A}\,(\Delta\ rule)$$

(related to the modality in S4)

Automated Generation of HC

The involved logics

- should admit some suitable rules
- their cut-free (hyper)sequent calculi have to have permulation rules, ”standard” rules for connectives...

Automated Generation of HC

(M. Baaz, *. ”Generating Inference Systems for logics with linearity”. In preparation)

Logic L

((single conclusioned) cut free (hyper)sequent calculus

Automated Generation of HC

Logic L

((single conclusioned) cut free (hyper)sequent calculus S

$$+$$

$$(A \to B) \vee (B \to A)$$

Automated Generation of HC

Logic L

((single conclusioned) cut free (hyper)sequent calculus S

$$+$$

$$(A \rightarrow B) \vee (B \rightarrow A)$$

$$\Longrightarrow \text{Logic } L^{\star}$$

Automated Generation of HC

Logic L

((single conclusioned) cut free (hyper)sequent calculus

$$+$$

$$(A \to B) \vee (B \to A)$$

$$\Longrightarrow \text{Logic } L^\star$$

(cut free hypersequent sequent calculus):

Automated Generation of HC

Logic L

((single conclusioned) cut free (hyper)sequent calculus

$$+$$

$$(A \to B) \vee (B \to A)$$

$\Longrightarrow$ Logic $L^\star$

(cut free hypersequent sequent calculus):

- hypersequent version of S
- $$\frac{G|\Gamma, \Gamma' \vdash A \quad G'|\Gamma_1, \Gamma_1' \vdash A'}{G|G'|\Gamma, \Gamma_1' \vdash A|\Gamma', \Gamma_1 \vdash A'} \ (com)$$

Example: Basic fuzzy logics

- MTL
- Urquhart’s C logic (versions I and II)

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- MTL
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Main formalization of fuzzy logic: Gödel , Łukasiewicz and Product logic .
See: (P. Hájek. ”Metamathematics of Fuzzy Logic”. Kluwer. 1998)

Example: Basic fuzzy logics

- MTL
- Urquhart's C logic (versions I and II)

Main formalization of fuzzy logic: Gödel , Łukasiewicz and Product logic . Correspond to the most important t-norms that are the main tool to combine fuzzy information.

By a t-norm one means some binary operation $*$ in the real unit interval $[0, 1]$ which is associative, commutative, non-decreasing in both arguments and which has 1 as a neutral element.

Example: Basic fuzzy logics

- MTL
- Urquhart's C logic (versions I and II)

Main formalization of fuzzy logic: Gödel $x * y = \min(x, y)$, Łukasiewicz $x * y = \max(0, x + y - 1)$ and Product logic $x * y = x \cdot y$.
Correspond to the most important t-norms that are the main tool to combine fuzzy information.

By a t-norm one means some binary operation $*$ in the real unit interval $[0, 1]$ which is associative, commutative, non-decreasing in both arguments and which has 1 as a neutral element.

Basic fuzzy logics: MTL

As well as defining logics based on particular t-norm, logics may also be based on classes of t-norms. (Logics are identified with the formulas valid in all the logics based on t-norms from that class)

Basic fuzzy logics: MTL

As well as defining logics based on particular t-norm, logics may also be based on classes of t-norms. (Logics are identified with the formulas valid in all the logics based on t-norms from that class) (F. Esteva and L. Godo. ”Monoidal t-norm based Logic: towards a logic for left-continuous t-norms”. Fuzzy Sets and Systems. 2001”)

MTL is the logic of *left-continuous t-norms*.

Basic fuzzy logics: MTL

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MTL is the logic of *left-continuous t-norms*.

$$\text{MTL} = \text{aMAILL} + (A \to B) \vee (B \to A)$$

Basic fuzzy logics: C

(A. Urquhart. ”Many-Valued Logic”. In: Handbook of Philosophical Logic, Vol. III. First Edition. 1986)

Basic fuzzy logics: C

Ax1. $A \to (B \to A)$

Ax2. $(A \to B) \to [(C \to A) \to (C \to B)]$

Ax3. $[A \to (C \to B)] \to [C \to (A \to B)]$

Ax4. $(A \wedge B) \to A$

Ax5. $(A \wedge B) \to B$

Ax6. $A \to [C \to (A \wedge C)]$

Ax7. $A \to (A \vee B)$

Ax8. $B \to (A \vee B)$

Ax9. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

Lin. $(A \to B) \vee (B \to A)$

Basic fuzzy logics: C

Urquhart claimed that C is semantically characterized by model structures on ordered commutative monoids.

Basic fuzzy logics: C

To obtain completeness in (A. Urquhart. ”Many-Valued Logic”. In: Handbook of Philosophical Logic, Vol. III. Second Edition. 2001) Urquhart added to C the following axioms:

U1. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$

U2. $((A^k \rightarrow C) \wedge (B^k \rightarrow C)) \rightarrow ((A \vee B)^k \rightarrow C)$ for every $k \geq 2$, where $A^k \rightarrow C$ stands for $\underbrace{A \rightarrow (A \rightarrow (\ldots (A}_{k \text{ times}} \rightarrow C) \ldots))$.

Basic fuzzy logics

Basic fuzzy logics

Hilbert system for IL <u>without contraction</u>

Basic fuzzy logics

Hilbert system for IL <u>without contraction</u> with the following axioms for AND

- $(A \wedge B) \rightarrow A \quad (A \wedge B) \rightarrow B$,
  $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$,
  $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \odot B) \rightarrow C)$,
  $A \rightarrow [C \rightarrow (A \odot C)]$
- $(A \wedge B) \rightarrow A$, $(A \wedge B) \rightarrow B$,
  $A \rightarrow [C \rightarrow (A \wedge C)]$

Basic fuzzy logics

Hilbert system for IL without contraction with the following axioms for AND + $(A \rightarrow B) \vee (B \rightarrow A)$

- $(A \wedge B) \rightarrow A \quad (A \wedge B) \rightarrow B$,
  $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$,
  $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \odot B) \rightarrow C)$,
  $A \rightarrow [C \rightarrow (A \odot C)]$: MTL
- $(A \wedge B) \rightarrow A$, $(A \wedge B) \rightarrow B$,
  $A \rightarrow [C \rightarrow (A \wedge C)]$: Urquart's C logic (version I)

HC for basic fuzzy logics

LJ <u>without contraction</u>

(Ono, H. and Komori, Y. ”Logics without the contraction rule ”. J. of Symbolic Logic. 1985)

HC for basic fuzzy logics

LJ without contraction

(Ono, H. and Komori, Y. ”Logics without the contraction rule ”. J. of Symbolic Logic. 1985)

$$(\wedge, r)_m \quad \frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B} \qquad (\wedge, l)_m \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}$$

$$(\wedge, r)_a \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad (\wedge_i, l)_a \quad \frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$

HC for Basic fuzzy logics

LJ without contraction

- both additive and multiplicative rules for AND (i.e. $\wedge$ and $\odot$)
- $(\wedge, r)_m \quad \dfrac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B}$

  $(\wedge_i, l)_a \quad \dfrac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$
- additive rules for AND

HC for Basic fuzzy logics

LJ without contraction

- both additive and multiplicative rules for AND (i.e. $\wedge$ and $\odot$) (calculus for aMAILL)
- $$\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B}$$
  $$\frac{\Gamma, A_i \vdash C}{\Gamma, A_1 \wedge A_2 \vdash C}$$
  (calculus for $C - (A \rightarrow B) \vee (B \rightarrow A)$)
- additive rules for AND

HC for Basic fuzzy logics

Hypersequent version of LJ without contraction +

$$\frac{G|\Gamma,\Gamma'\vdash A \quad G'|\Gamma_1,\Gamma_1'\vdash A'}{G|G'|\Gamma,\Gamma_1'\vdash A|\Gamma',\Gamma_1\vdash A'}\ (com)$$

- MTL: both additive and multiplicative rules for AND (i.e. $\wedge$ and $\odot$)
- Urquart's C logic I: $$\frac{G|\Gamma\vdash A \quad G'|\Gamma'\vdash B}{G|G'|\Gamma,\Gamma'\vdash A\wedge B}$$

$$\frac{G|\Gamma, A_i\vdash C}{G|\Gamma, A_1\wedge A_2\vdash C}$$

- Urquart's C logic II: additive rules for AND

Urquart’s C logic (version II)

Urquhart defined the new C logic by adding to C the following axioms:

U1. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

U2. $((A^k \to C) \wedge (B^k \to C)) \to ((A \vee B)^k \to C)$ for every $k \geq 2$, where $A^k \to C$ stands for $\underbrace{A \to (A \to (\ldots (A}_{k \text{ times}} \to C) \ldots))$.

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Fact 1: the hypersequent version of LJ without contraction with the additive rules for AND characterize the logic $C + U1$.

Urquart's C logic (version II)

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Fact 1: the hypersequent version of LJ without contraction with the additive rules for AND characterize the logic $C + U1$.
Fact 2: Axiom U2 is derivable in this calculus
(*, "On Urquhart's C logic". Proc. of ISMVL'2000)

Urquart’s C logic (version II)

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Fact 1: the hypersequent version of LJ without contraction with the additive rules for AND characterize the logic $C + U1$.
Fact 2: Axiom U2 is derivable in this calculus

Corollary

- The above hypersequent calculus characterize the logic C (version II)
- Axiom U2 is redundant in the axiomatization of C (version II)

Automated Generation of HC with $\triangle$

Logic L

((single conclusioned) cut free sequent calculus S)

Automated Generation of HC with $\triangle$

Logic L
((single conclusioned) cut free sequent calculus S)
$+$ the modality Δ

$$(1)\ \Delta A \rightarrow A \qquad (2)\ \Delta A \rightarrow \Delta\Delta A$$
$$(3)\ \Delta A \vee \neg\Delta A \quad (4)\ \Delta(A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)$$

$$\frac{A}{\Delta A}\ (\Delta\ rule)$$

(if L is a first order logic:

$$(5)\ \forall x \Delta A(x) \rightarrow \Delta\forall x A(x))$$

Automated Generation of HC with $\triangle$

Logic L
((single conclusioned) cut free sequent calculus S)
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$\Longrightarrow$ Logic L^{Δ}

Automated Generation of HC with Δ

Logic L
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(cut free hypersequent sequent calculus):

Automated Generation of HC with Δ

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(cut free hypersequent sequent calculus):

- hypersequent version of S
-

$$\frac{G \mid \Gamma, A \vdash C}{G \mid \Gamma, \Delta A \vdash C}\ (\Delta, l) \qquad \frac{G \mid \Delta\Gamma \vdash A}{G \mid \Delta\Gamma \vdash \Delta A}\ (\Delta, r)$$

$$\frac{G \mid \Delta\Gamma, \Gamma' \vdash A}{G \mid \Delta\Gamma \vdash \ \mid \Gamma' \vdash A}\ (cl_{\Delta}, l)$$

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = FO Intuitionistic Logic + Δ
  G. Takeuti, S. Titani. ”Globalization of intuitionistic set theory.” Annals of Pure and Applied Logic. 1987

- Global Intuitionistic Fuzzy Logic = FO Gödel Logic + Δ
  G. Takeuti, S. Titani. “Global Intuitionistic Fuzzy Set Theory.” In: The Mathematics of Fuzzy Systems. 1986

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = FO Intuitionistic Logic + Δ
  Sequent calculus: obtained by adding to (a suitable modification of) Maehara's LJ' the rules for $\Box$ of the sequent calculus for the modal logic **S5**, i.e.

  $$\frac{\Gamma, A \vdash \Sigma}{\Gamma, \Delta A \vdash \Sigma}\ (\Box_l)' \qquad\qquad \frac{\overline{\Gamma} \vdash A, \overline{\Sigma}}{\overline{\Gamma} \vdash \Delta A, \overline{\Sigma}}\ (\Box_r)'$$

  (where $\overline{\Sigma}$ and $\overline{\Gamma}$ are sets of Δ-closed formulas)

Ex: Global Int (Fuzzy) Logic

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  Sequent calculus: obtained by adding to (a suitable modification of) Maehara's LJ' the rules for $\Box$ of the sequent calculus for the modal logic **S5**, LJ' is an equivalent version of Gentzen's LJ where the restriction to at most one formula in the succedent of sequents applies not generally but only in the case of the right rules for $\longrightarrow$, $\neg$ and $\forall$. E.g., the $\longrightarrow$ right rule is

$$\frac{\Gamma, A \vdash B, (\overline{\Sigma})}{\Gamma \vdash A \to B, (\overline{\Sigma})}\ (\to, r)'$$

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = FO Intuitionistic Logic + Δ
  Sequent calculus: obtained by adding to (a suitable modification of) Maehara's LJ' the rules for $\Box$ of the sequent calculus for the modal logic **S5**,
  This calculus does not admit elimination of cuts. (E.g. the sequent

  $$\vdash \forall x \neg \Box A(x), \exists x A(x)$$

  is derivable in this calculus *only* using cuts.)

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
  Cut free Hypersequent Calculus = Hypersequent version of LJ with in addition

$$\frac{G \mid \Gamma, A \vdash C}{G \mid \Gamma, \Delta A \vdash C}\ (\Delta, l) \qquad \frac{G \mid \Delta\Gamma \vdash A}{G \mid \Delta\Gamma \vdash \Delta A}\ (\Delta, r)$$

$$\frac{G \mid \Delta\Gamma, \Gamma' \vdash A}{G \mid \Delta\Gamma \vdash \ \mid \Gamma' \vdash A}\ (cl_\Delta, l)$$

- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta
  Cut free Hypersequent Calculus = Hypersequent version of LJ with in addition (Δ, l), (Δ, r), (cl_Δ, l) and (com).

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta

In both cases, the hypersequent calculus allows one to prove a (suitable version of the) midsequent theorem.

Ex: Global Int (Fuzzy) Logic

- Global Intuitionistic Logic = Intuitionistic Logic + Delta
- Global Intuitionistic Fuzzy Logic = Gödel Logic + Delta

In both cases, the hypersequent calculus allows one to prove a (suitable version of the) midsequent theorem.

(For GI it holds for proofs in which $(\vee, l)$ is applied only to formulas where (at least) one of the disjuncts is a Δ-formula (i.e. $\Delta A \vee B$))

*, ”A proof-theoretical investigation of global intuitionistic (fuzzy) logic”. Draft. 2003

Variants of the Hypersequent Framework

Variants of the Hypersequent Framework

”simple” disjunctive conditions

Variants of the Hypersequent Framework

E.g.,

$$IL +$$

$$\begin{array}{cc} \neg A \vee A & (Classical\ logic) \\ (A \rightarrow B) \vee (B \rightarrow A) & (Goedel\ logic) \\ \neg A \vee \neg\neg A & (LQ) \end{array}$$

Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models

- Łukasiewicz Logic

Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models : IL + the axiom scheme (Bd_k) recursively defined as follows:

$$\begin{array}{rl} (Bd_1) & A_1 \vee \neg A_1 \\ (Bd_{i+1}) & A_{i+1} \vee (A_{i+1} \rightarrow (Bd_i)) \end{array}$$

- Łukasiewicz Logic : aMALL + $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$

Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models : IL + the axiom scheme (Bd_k) recursively defined as follows:

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- Łukasiewicz Logic : aMALL + $((A \longrightarrow B) \longrightarrow B) \longrightarrow ((B \longrightarrow A) \longrightarrow A)$

$$A \vee B = \min\{A, B\} \qquad A \wedge B = \max\{A, B\}$$

$$A \oplus B = \min\{1, A + B\} \qquad A \odot B = \max\{0, A + B$$

$$A \longrightarrow B = \min\{1, A + B - 1\} \qquad \neg A = 1 - A$$

Variants of the Hypersequent framework

- Intermediate Logics of bounded depth Kripke models : IL + the axiom scheme (Bd_k) recursively defined as follows:

$$\begin{array}{rl} (Bd_1) & A_1 \vee \neg A_1 \\ (Bd_{i+1}) & A_{i+1} \vee (A_{i+1} \rightarrow (Bd_i)) \end{array}$$

- Łukasiewicz Logic : aMALL + $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$ allows one to define the additive connectives $\wedge$ and $\vee$ over the multiplicative ones $\odot$ and $\oplus$. (E.g. $A \wedge B = A \odot (A \rightarrow B)$)

HC for Logics of Kripke models with depth $\leq k$

Idea: introduce additional structure on hypersequents

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Elimination of the (EE) rule.

(a hypersequent in this framework is a suitable ordered sequence of components)

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Analytic calculus for these logics = (Suitable) ”hypersequent” calculus for IL + a single uniform rule (for each k).

HC for Logics of Kripke models with depth $\leq k$

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Elimination of the (EE) rule.
(a hypersequent in this framework is a suitable ordered sequence of components)
Analytic calculus for these logics = (Suitable) ”hypersequent” calculus for IL + a single uniform rule (for each k).

*, M. Ferrari. ” Hypertableau and Path-Hypertableau Calculi for some families of intermediate logics”. Tableaux 2000.

HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculi are defined by dualizing hypersequent calculi: Given a hypersequent $\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$, each component $\Gamma_i \vdash \Delta_i$ translates into the set of signed formulas $\mathbf{T}(\Gamma_i) \cup \mathbf{F}(\Delta_i)$ where $\mathbf{T}(\Gamma_i) = \{\mathbf{T}A \mid A \in \Gamma_i\}$ and $\mathbf{F}(\Delta_i) = \{\mathbf{F}A \mid A \in \Delta_i\}$. Thus the above hypersequent translates into the *h-set*

$$\mathbf{T}(\Gamma_1) \cup \mathbf{F}(\Delta_1) \mid \ldots \mid \mathbf{T}(\Gamma_n) \cup \mathbf{F}(\Delta_n).$$

HC for Logics of Kripke models with depth $\leq k$

By dualizing hypersequent calculi one can define tableau calculi as follows: Given a hypersequent $\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$, each component $\Gamma_i \vdash \Delta_i$ translates into the set of signed formulas $\mathbf{T}(\Gamma_i) \cup \mathbf{F}(\Delta_i)$ where $\mathbf{T}(\Gamma_i) = \{\mathbf{T}A \mid A \in \Gamma_i\}$ and $\mathbf{F}(\Delta_i) = \{\mathbf{F}A \mid A \in \Delta_i\}$. Thus the above hypersequent translates into the *h-set*

$$\mathbf{T}(\Gamma_1) \cup \mathbf{F}(\Delta_1) \mid \ldots \mid \mathbf{T}(\Gamma_n) \cup \mathbf{F}(\Delta_n).$$

We say that an *h-set* $S_1 \mid \ldots \mid S_n$ is *realized* in $\underline{K}$, if all the sets S_j, with $j = 1, \ldots, n$, are realized in $\underline{K}$ (Given a Kripke model $\underline{K} = \langle P, \leq, v \rangle$ and a signed formula H, we say that $\alpha \in P$ *realizes* H if $H \equiv \mathbf{T}X$ and $\alpha \Vdash X$, or $H \equiv \mathbf{F}X$ and $\alpha \nVdash X$.)

HC for Logics of Kripke models with depth $\leq k$

Ex. (Hyper)Tableau calculus for IL

External Structural Rules

$$\frac{\Psi \mid \Phi}{\Psi}\,\mathrm{HEW} \qquad \frac{\Psi \mid S}{\Psi \mid S \mid S}\,\mathrm{HEC} \qquad \frac{\Psi \mid S_1 \mid S_2 \mid \Phi}{\Psi \mid S_2 \mid S_1 \mid \Phi}\,\mathrm{HEE}$$

Logical Rules

$$\frac{\Psi \mid S, \mathbf{T}(A_1 \wedge A_2)}{\Psi \mid S, \mathbf{T}A_i}\,\mathbf{T}\wedge_{\mathrm{i}} \quad \text{for } i = 1, 2 \qquad \frac{\Psi \mid S, \mathbf{F}(A \wedge B)}{\Psi \mid S, \mathbf{F}A \parallel \Psi \mid S, \mathbf{F}B}\,\mathbf{F}\wedge$$

$$\frac{\Psi \mid S, \mathbf{T}(A \vee B)}{\Psi \mid S, \mathbf{T}A \parallel \Psi \mid S, \mathbf{T}B}\,\mathbf{T}\vee \qquad \frac{\Psi \mid S, \mathbf{F}(A_1 \vee A_2)}{\Psi \mid S, \mathbf{F}A_i}\,\mathbf{F}\vee_{\mathrm{i}} \quad \text{for } i = 1, 2$$

$$\frac{\Psi \mid S, \mathbf{T}(A \rightarrow B)}{\Psi \mid S, \mathbf{T}(A \rightarrow B), \mathbf{F}A \parallel \Psi \mid S, \mathbf{T}B}\,\mathbf{T}\rightarrow \qquad \frac{\Psi \mid S, \mathbf{F}(A \rightarrow B)}{\Psi \mid S^{\mathbf{T}}, \mathbf{T}A, \mathbf{F}B}\,\mathbf{F}\rightarrow$$

$$S^{\mathbf{T}} = \{\mathbf{T}X \mid \mathbf{T}X \in S\}$$

HC for Logics of Kripke models with depth $\leq k$

New interpretation: We say that an h-set $S_1 \mid \ldots \mid S_n$ is *path-realized* in a Kripke model $\underline{K}$, if there exists a path $\underline{\alpha} = \alpha_1, \ldots, \alpha_n \in \underline{K}$ $(\alpha_1 \leq \ldots \leq \alpha_n)$ such that α_i realizes S_i, for every $1 \leq i \leq n$.

HC for Logics of Kripke models with depth $\leq k$

We say that an h-set $S_1 \mid \ldots \mid S_n$ is *path-realized* in a Kripke model $\underline{K}$, if there exists a path $\underline{\alpha} = \alpha_1, \ldots, \alpha_n \in \underline{K}$ $(\alpha_1 \leq \ldots \leq \alpha_n)$ such that $\alpha_i \triangleright S_i$, for every $1 \leq i \leq n$.

In the new interpretation of h-sets the external exchange rule does not hold.

HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculus for IL became External Structural Rules:

$$\frac{\Psi \mid \Phi}{\Psi}\,\mathrm{EC_r} \qquad \frac{\Psi \mid \Phi}{\Phi}\,\mathrm{EC_l}$$

$$\frac{\Psi \mid \Phi}{\Psi \mid \Phi \mid \Phi}\,\mathrm{EW_r} \qquad \frac{\Psi \mid \Phi}{\Psi \mid \Psi \mid \Phi}\,\mathrm{EW_l}$$

HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculus for IL became
Logical Rules
E.g.

$$\frac{\Psi \mid S, \mathbf{T}(A_1 \wedge A_2) \mid \Psi'}{\Psi \mid S, \mathbf{T}A_i \mid \Psi'}\,\mathbf{T}\wedge_{\mathrm{i}}$$

HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculus for IL became
Logical Rules
But

$$\frac{\Psi \mid S, \mathbf{F}(A \to B) \mid {\color{red}\Psi'}}{\Psi \mid S^{\mathbf{T}}, \mathbf{T}A, \mathbf{F}B}\,\mathbf{F}{\to}$$

HC for Logics of Kripke models with depth $\leq k$

(Hyper)Tableau calculus for IL became
Logical Rules

$$+$$

From $\Psi \mid S_0 \mid \ldots \mid S_k \mid \Psi'$ one derives $\Psi \mid S_0, S_1 \mid \Psi' \parallel \Psi \mid S_0 \mid S_1, S_2 \mid \Psi' \parallel \ldots \parallel \Psi \mid S_0 \mid$ $\ldots \mid S_{k-2} \mid S_{k-1}, S_k \mid \Psi'$.

HC for Łukasiewicz Logic

HC for Łukasiewicz Logic

- Infinite-valued Łukasiewicz logic is one of the main formalizations of Fuzzy Logic (Hajék 1998)
- Formulae in Łukasiewicz logic stand to particular geometric functions as formulae in classical logic stand to boolean functions. (McNaughton 1951)
- Łukasiewicz Logic(s) formalise Ulam's game (a variant of the game of Twenty Questions where errors/lies are allowed in the answers) (Mundici 1992)

HC for Łukasiewicz Logic

R. Hähnle. ”Many-Valued Logic and Mixed Integer Programming”. Annals of Math. and Artificial Intell. 1994. (solving integer programs)

D. Mundici, N. Olivetti. ”Resolution and model building in the infinite-valued calculus of Łukasiewicz”. TCS. 1998. (intersecting hyperplanes)

H. Wagner. ”A new resolution calculus for the infinite-valued propositional logic of Łukasiewicz”. J of Applied Non-Classical Logics. 2000. (determining θ-supports of formulas)

S. Aguzzoli, *. ”Finiteness in infinite-valued Łukasiewicz logic”. J. of Logic Language and Information. 2000. (reductions to finite-valued Łukasiewicz Logics)

HC for Łukasiewicz Logic

3-valued Łukasiewicz Logic
aMALL + $\neg A \vee A \oplus A$

hypersequent calculus for aMALL +

$$\frac{G|\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3 \quad G'|\Gamma'_1, \Gamma'_2, \Gamma'_3 \vdash \Delta'_1, \Delta'_2, \Delta}{G|G'|\Gamma_1, \Gamma'_1 \vdash \Delta_1, \Delta'_1|\Gamma_2, \Gamma'_2 \vdash \Delta_2, \Delta'_2|\Gamma_3, \Gamma'_3 \vdash \Delta_3, \Delta}$$

(A. Avron. ”Natural 3-valued Logics. Characterization and Proof Theory”. JSL. 1991.) or

$$\frac{G|\Sigma, \Gamma_1 \vdash \Delta_1, \Pi \quad G'|\Sigma, \Gamma_2 \vdash \Delta_2, \Pi}{G|G'|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2|\Sigma \vdash \Pi}$$

(*, D.M. Gabbay, N. Olivetti. ”Cut-free proof systems for logics of weak excluded middle”. Soft Computing, 1998.)

HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic

aMALL + $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$

HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic

aMALL + $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$

G. Metcalfe, N. Olivetti, D. Gabbay. Sequent and Hypersequent Calculi for Abelian and Łukasiewicz Logics. Submitted. 2003

HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic

aMALL + $((A \to B) \to B) \to ((B \to A) \to A)$

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Note: the interpretation of components (i.e. sequents) is changed!

HC for Łukasiewicz Logic

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Note: the interpretation of components (i.e. sequents) is changed!

$\Gamma \vdash \Delta$ is <u>not</u> interpreted as $(AND\ A \in)\Gamma \to (OR\ B \in)\Delta$ but using the characteristic model $[-1, 0]$ of Łukasiewicz Logic.

HC for Łukasiewicz Logic

Infinite-valued Łukasiewicz Logic
aMALL + $((A \to B) \to B) \to ((B \to A) \to A)$

G. Metcalfe, N. Olivetti, D. Gabbay. Sequent and Hypersequent Calculi for Abelian and Łukasiewicz Logics. Submitted. 2003

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I.e. if for all valuations v for $[-1, 0]$,

$$\Sigma_{A\in\Gamma} v(A) \leq \Sigma_{B\in\Delta} v(B)$$

(where $\Sigma_{A\in\emptyset} v(A) = 0$)

HC for Łukasiewicz Logic

$$(ID)\quad A \vdash A \qquad\qquad (\Lambda)\quad \vdash \qquad\qquad (\bot)\quad \bot \vdash A$$

Logical rules

$$\frac{G|\Gamma, B \vdash A, \Delta|\Gamma \vdash \Delta}{G|\Gamma, A \to B \vdash \Delta} \qquad \frac{G|\Gamma \vdash \Delta \quad G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \to B, \Delta}$$

Internal structural rules

$$\frac{G|\Gamma \vdash \Delta}{G|\Gamma, A \vdash \Delta}\,(WL) \qquad \frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,(M)$$

External structural rules

(EW), (EE), (EC) and $$\frac{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G|\Gamma_1 \vdash \Delta_1|\Gamma_2 \vdash \Delta_2}\,(S)$$

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HC are a nice tool for analyzing and reasoning about proofs in the logics concerned.